Finite-time output regulation for homogeneous quasilinear hyperbolic systems

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Abstract

This work is concerned with the finite-time output regulation problem for homogeneous quasilinear hyperbolic systems with one-side controls and with nonlinear boundary condition at the other side. The to-be-controlled outputs are the boundary outputs of the uncontrolled components on the same side as the controls. The reference signal is assumed to be priori known and its derivative is assumed to have compact support. We employ a time-dependent feedback regulator as the control to achieve the output regulation for nonlinear systems. For sufficiently small initial data and reference signal, the output regulation problem is solved under the same condition for the exact controllability of hyperbolic systems by one-side boundary controls. The resulting feedback control formally depends on future system state and introduces non-local boundary conditions to the closed-loop system. This brings new difficulties in proving the local well-posedness of quasilinear systems.

Keywords: output regulation, hyperbolic systems, nonlinear systems

MSC Classification: 93B52 , 93C20 , 35L50

1 Introduction and problem statement

The control of hyperbolic systems is of fundamental importance in both mathematical theory and engineering applications. These systems arise in diverse fields such as traffic flow modeling and gas flow pipelines. For extensive examples of hyperbolic systems in diverse applications, we refer to [1] and the references therein. In this paper, we consider the finite-time output regulation problem for the following homogeneous quasilinear hyperbolic system

$$\partial_t w(t,x) + \Lambda(x, w(t,x))\partial_x w(t,x) = 0, \quad t \geq 0,\ 0 \leq x \leq 1, \tag{1}$$

where $w = (w_1, \cdots, w_n)^\top : [0,\infty) \times [0,1] \to \mathbb{R}^n$, and $\Lambda : [0,1] \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a real matrix-valued function. Let us make the following assumptions for Λ.

Assumption 1 *The matrix Λ is diagonal and has $m \geq 1$ distinct negative eigenvalues and $p = n - m \geq 1$ distinct positive eigenvalues, namely for every (x,y) in $[0,1] \times \mathbb{R}^n$,*

$$\Lambda(x,y) = \operatorname{diag}(\lambda_1(x,y), \cdots, \lambda_m(x,y), \lambda_{m+1}(x,y), \cdots, \lambda_n(x,y)),$$

where

$$\lambda_1(x,w) < \cdots < \lambda_m(x,w) < 0 < \lambda_{m+1}(x,w) < \cdots < \lambda_n(x,w).$$

Function λ_i is in $C^2([0,1] \times \mathbb{R}^n)$ for $1 \leq i \leq n$.

All along this paper, for a vector (or vector-valued function) ν in $\mathbb{R}^n$ and a matrix (or matrix-valued function) B in $\mathbb{R}^{n\times n}$, we use the notation

$$\nu = \begin{pmatrix} \nu_- \\ \nu_+ \end{pmatrix}, \quad B = \begin{pmatrix} B_{--} & B_{-+} \\ B_{+-} & B_{++} \end{pmatrix},$$

with ν_- in $\mathbb{R}^m$, ν_+ in $\mathbb{R}^p$ and B_{--} in $\mathbb{R}^{m\times m}$, B_{-+} in $\mathbb{R}^{m\times p}$, B_{+-} in $\mathbb{R}^{p\times m}$, B_{++} in $\mathbb{R}^{p\times p}$. The following types of boundary conditions and controls are considered. The boundary condition at $x = 0$ is given by

$$w_+(t,0) = Q(w_-(t,0)), \quad t \geq 0, \tag{2}$$

with function $Q : \mathbb{R}^m \to \mathbb{R}^p$ satisfying the following assumptions.

Assumption 2 *Function Q is in $C^2(\mathbb{R}^m)^p$ with $Q(0) = 0$.*

The boundary control at $x = 1$ is

$$w_-(t,1) = u(t) = (u_1, \cdots, u_m)^\top(t), \quad t \geq 0. \tag{3}$$

Dedicated to Jean-Michel Coron, on the occasion of his 70th birthday.

The initial condition is given by

$$w(0,x) = w^0(x), \quad 0 \le x \le 1, \tag{4}$$

where $w^0 = (w_1^0, \cdots, w_n^0)^\top : [0,1] \to \mathbb{R}^n$ satisfies the following assumptions.

Assumption 3 *Initial data w^0 is in $C^1([0,1])^n$ and satisfies the compatibility conditions*

$$w_+^0(0) = Q(w_-^0(0)), \quad \Lambda_{++}(0, w^0(0))\partial_x w_+^0(0) = \nabla Q(w_-^0(0))\Lambda_{--}(0, w^0(0))\partial_x w_-^0(0). \tag{5}$$

Assumptions 1 to 3 are consistent with those in [2]. They are usual assumptions for considering the local C^1 solutions of quasilinear hyperbolic systems.

In this work, we concern the finite-time output regulation problem, namely designing a feedback regulator such that the output of the system tracks the given reference signal in finite time. The following to-be-controlled output and reference signal are considered. The to-be-controlled output is given by

$$y(t) = w_+(t,1). \tag{6}$$

The reference signal $r = (r_1, \cdots, r_p)^\top : [0,\infty) \to \mathbb{R}^p$ is assumed to be priori known and satisfies the following assumptions.

Assumption 4 *The reference signal r is in $C^1([0,\infty))^p$. The derivative r' has compact support in $[0,\infty)$.*

Assumption 4 will be commented in the item 3 of Remark 1. Denote by

$$e_y(t) = y(t) - r(t) \tag{7}$$

the output tracking error. Let us give the notion of the regulation that we are interested in.

Definition 1 Let $T > 0$. The output y of system (1)-(4) and (6) achieves the finite-time output regulation within settling time T, if there exist $\varepsilon > 0$ and a feedback regulator u such that for all initial states w^0 in $C^1([0,1])^n$ satisfying (5) and $\|w^0\|_{C^1([0,1])^n} < \varepsilon$, and all reference signals r in $C^1([0,\infty))^p$ satisfying Assumption 4 and $\|r\|_{C^1([0,\infty))^p} < \varepsilon$, the output tracking error e_y satisfies $e_y(t) = 0$ for $t \ge T$.

Over the past few decades, research on control problems for quasilinear hyperbolic systems has become substantially enriched, particularly regarding stabilization problems. To the best of our knowledge, the pioneer works are the studies in [3, 4] on homogeneous 2×2 quasilinear hyperbolic systems. A generalization to homogeneous $n \times n$ systems was given by [5–7]. Particularly, [8] establishes the theory on semi-global classical solutions to general nonautonomous quasilinear hyperbolic systems and applies it to controllability problems. All these results rely on a systematic use of direct estimates of the solutions and their derivatives along the characteristic curves. Moreover, all these results provide decay estimates in the C^1-norm. For decay estimates in the H^2-norm for quasilinear hyperbolic systems, [9] addressed the case of homogeneous 2×2 systems, and subsequently, [10] dealt with homogeneous $n \times n$ systems. The methods used in [9, 10] are both based on the direct Lyapunov method. The key point is that the energy-like Lyapunov functions for the H^2-norm are easier to handle. Nevertheless, [11] used an energy-like Lyapunov function for the C^1-norm to give an alternative proof to a result that had already been shown with a characteristic approach (see [5–7]). For the boundary stability of inhomogeneous hyperbolic systems, one can refer to [1, 12, 13]. Another interesting topic is the finite-time stabilization problem for hyperbolic systems. One can refer to [14] for the finite-time stabilization of 2×2 quasilinear hyperbolic systems. The finite-time stabilization in optimal time of quasilinear system was investigated in [2]. Later on, [15] presented Lyapunov functions for the feedbacks in [2] and used estimates for Lyapunov functions to rediscover the finite stabilization results.

This paper concerns the finite-time output regulation problem. To the best of our knowledge, existing literature has only addressed output regulation for linear hyperbolic systems and linear hyperbolic systems coupled with nonlinear ODEs, while the corresponding problem for quasilinear hyperbolic systems remains unexplored. For the output regulation problems, unlike the stabilization problems, the objective is to design feedback regulator such that the output of the system tracks a given reference and rejects the disturbances. The first result on the finite-time output regulation for hyperbolic systems was obtained in [16], where the backstepping method was used to design the feedback regulator for boundary controlled linear 2×2 time-invariant hyperbolic systems. Later on, [17, 18] achieved finite-time output regulation for general $n \times n$ time-invariant hyperbolic systems with different convergent time. Moreover, [19] solved finite-time output regulation for time-varying linear hyperbolic systems. Concerning the output regulation problem for nonlinear systems, [20] considered the output regulation problem for boundary-controlled linear hyperbolic PDEs that are bidirectionally coupled with nonlinear ODEs.

In this paper, we consider the finite-time output regulation problem for homogeneous quasilinear hyperbolic systems as defined in Definition 1. Set

$$\tau_i = \int_0^1 \frac{1}{|\lambda_i(x,0)|} \mathrm{d}x, \quad 1 \le i \le n. \tag{8}$$

The main result of this paper is the following theorem.

Theorem 5 *Let Λ, Q, w^0 and r satisfy Assumptions 1 to 4. Assume that*

$$\operatorname{rank}(\nabla Q(0)) = p. \tag{9}$$

Let

$$T_0 = \tau_m + \tau_{m+1}. \tag{10}$$

For any $T > T_0$, set $\tau_0 = T - T_0$. There exist $\varepsilon = \varepsilon(\tau_0) > 0$ and a feedback regulator

$$u(t) = H(t, w(t + \cdot, \cdot), r, w^0), \quad t \geq 0, \tag{11}$$

such that if $\|w^0\|_{C^1([0,1])^n} < \varepsilon$ and $\|r\|_{C^1([0,\infty))^p} < \varepsilon$, then there exists a unique solution w in $C^1([0,\infty) \times [0,1])^n$ to system (1)-(4) with feedback regulator (11). The following estimate holds for some positive constant C independent of w^0, r, τ_0 and ε,

$$\|w\|_{C^1([0,\infty)\times[0,1])^n} \leq C(1 + \tau_0^{-1})(\|w^0\|_{C^1([0,1])^n} + \|r\|_{C^1([0,\infty))^p}). \tag{12}$$

Moreover, The output y of system (1)-(4) and (6) achieves the finite-time output regulation within settling time T.

Remark 1

1. In Theorem 5 and what follows, $w(t + \cdot, \cdot)$ denotes the function $(s, x) \mapsto w(t + s, x)$. The non-local boundary control involving $w(t + \cdot, \cdot)$ is also considered in [2].
2. It follows from assumption (9) that $m \geq p$, which means that the number of controls is greater than or equal to the number of outputs. This full-row-rank condition is a classical requirement for the exact controllability of hyperbolic systems. Specifically, it serves as a necessary condition for the exact boundary controllability of both linear [21] and quasilinear [22, 23] hyperbolic systems by one-sided boundary controls.
3. Assumption 4 is due to the machinery of proof of the well-posedness of system (1)-(4). See Section 3.2 for details. Roughly speaking, Assumption 4 implies that $r'(t) = 0$ for sufficiently large t, which allows us to apply the Ascoli theorem in the well-posedness proof.
4. From the solution estimates (12), it can be observed that as the settling time T tends to T_0 (i.e., $\tau_0 \to 0$), the a priori estimate constant $C(1+\tau_0^{-1})$ tends to infinity. However, as shown in the proof of well-posedness (specifically in the Step 1 of the proof of Lemma 6), the C^1-norm estimate of the solution does not depend on τ_0. This leads to the admissible initial data and reference signals tending to zero as $\tau_0 \to 0$, namely
$$\varepsilon = \varepsilon(\tau_0) \to 0, \quad \text{as } \tau_0 \to 0.$$

Compared with existing results on finite-time output regulation for hyperbolic systems, particularly [16–18] on linear time-independent systems and [19] on linear time-varying systems, this paper investigates the output regulation problem for quasilinear hyperbolic systems. It is worth noting that, in contrast to [19], we have $e_y(t) = 0$ for all $t \geq T$ instead of $e_y(t) = 0$ for t in $[T, T_+]$ with some $T_+ > T$ as in [19]. The reason lies in our simpler system configurations. Unlike the general to-be-controlled output presented in [19], which encompasses distributed, boundary, and pointwise internal outputs, this work considers a specific case where the to-be-controlled output involves only boundary outputs. Furthermore, our model assumes the absence of source terms and external disturbances. These system configurations guarantee that, even when nonlinearities are taken into account, we are still able to obtain $e_y(t) = 0$ for all $t \geq T$.

The nonlinear nature of the system introduces new challenges to this problem. To achieve output regulation on nonlinear systems, we employ time-dependent feedback regulator. Following classical output regulation methodology, we obtain the required feedback regulator by solving time-varying regulator equations. This feedback regulator formally requires knowledge of the value of the reference signal at a future time instant.

The determination of this future time instant differs fundamentally between quasilinear and linear cases. For output regulation of linear hyperbolic systems, whether autonomous or non-autonomous, since the regulator equations depend solely on system parameters, this time instant is known a priori. However, in the quasilinear case, as the propagation speeds depend on the system state, the required time instant consequently becomes state-dependent. Formally speaking, this makes the time instant dependent on the future system state, which in turn means the resulting feedback regulator formally depends on future system state. In reality, after successfully proving the well-posedness of the closed-loop system, we demonstrate that the feedback regulator actually depends only on the current system state. Additionally, this feedback regulator introduces non-local boundary conditions to the closed-loop system, creating further difficulties in proving its well-posedness.

For establishing the well-posedness of the closed-loop system, we adapt the classical iterative methods from [2, 22]. Notably, when applying the Ascoli theorem to prove that the iteration sequence converges to a C^1 limit, we require the iterative functions to be defined on finite time intervals. Here we employ the assumption that the reference signal has compact support.

The remaining part of this paper is organized as follows. In Section 2, we introduce some preliminaries needed in the paper. We prove the main result, namely Theorem 5, in Section 3. Section 4 presents the conclusions and some perspectives. Appendix A provides the precise dynamics of two functions, which are employed in the design of feedback regulator.

2 Preliminaries

In this section, we provide some known facts on the characteristics associated to quasilinear hyperbolic systems and some properties for the coefficients involved in system (1)-(4).

2.1 Preliminaries on characteristics

Let us introduce the characteristics associated to system (1) and the entry and exit times as in [2, 19, 24]. For $t \geq 0$, $0 \leq x \leq 1$, $1 \leq i \leq n$ and ϕ in $C^1([0,\infty) \times [0,1])^n$ satisfying $\|\phi\|_{C^1([0,\infty)\times[0,1])^n} < \infty$, let $\chi_i^\phi(\cdot, t, x)$ denote the C^1 maximal solution to the problem

$$\frac{\mathrm{d}}{\mathrm{d}s}\chi_i^\phi(s;t,x) = \lambda_i(\chi_i^\phi(s;t,x), \phi(s, \chi_i^\phi(s;t,x))), \quad \chi_i^\phi(t;t,x) = x.$$

Here and in what follows, for $1 \leq i \leq n$, $\chi_i^\phi(\cdot, t, x)$ is defined on a certain subinterval $[s_i^{\mathrm{in},\phi}(t,x), s_i^{\mathrm{out},\phi}(t,x)]$ of $[0,\infty)$. Let us give the definition of the entry and exit times $s_i^{\mathrm{in},\phi}(t,x)$ and $s_i^{\mathrm{out},\phi}(t,x)$ associated to the characteristics $\chi_i^\phi(\cdot, t, x)$. For (t,x)

in $[0,\infty)\times[0,1]$, define the exit time $s_i^{\mathrm{out},\phi}(t,x)$ by the solution to

$$\begin{aligned}
&\chi_i^\phi(s_i^{\mathrm{out},\phi}(t,x);t,x)=0, \quad 1\le i\le m,\\
&\chi_i^\phi(s_i^{\mathrm{out},\phi}(t,x);t,x)=1, \quad m+1\le i\le n.
\end{aligned} \tag{13}$$

Introduce the sets for $1\le i\le m$,

$$\begin{aligned}
\mathcal{I}_i^\phi=&\{(t,x)\in[0,\infty)\times[0,1]:t\ge s_i^{\mathrm{out},\phi}(0,1)\\
&\qquad\text{or } 0\le t< s_i^{\mathrm{out},\phi}(0,1),\ \chi_i^\phi(t;0,1)<x\le 1\},\\
\mathcal{J}_i^\phi=&\{(t,x)\in[0,\infty)\times[0,1]:0\le t< s_i^{\mathrm{out},\phi}(0,1),\ 0\le x\le\chi_i^\phi(t;0,1)\},
\end{aligned} \tag{14}$$

and for $m+1\le i\le n$,

$$\begin{aligned}
\mathcal{I}_i^\phi=&\{(t,x)\in[0,\infty)\times[0,1]:t\ge s_i^{\mathrm{out},\phi}(0,0)\\
&\qquad\text{or } 0\le t< s_i^{\mathrm{out},\phi}(0,0),\ 0\le x<\chi_i^\phi(t;0,0)\},\\
\mathcal{J}_i^\phi=&\{(t,x)\in[0,\infty)\times[0,1]:0\le t< s_i^{\mathrm{out},\phi}(0,0),\ \chi_i^\phi(t;0,0)\le x\le 1\}.
\end{aligned} \tag{15}$$

Define the entry time $s_i^{\mathrm{in},\phi}(t,x)$ by

$$s_i^{\mathrm{in},\phi}(t,x)=\begin{cases}
\text{solution to } \chi_i^\phi(s_i^{\mathrm{in},\phi}(t,x);t,x)=1, & 1\le i\le m,\ (t,x)\in\mathcal{I}_i^\phi,\\
\text{solution to } \chi_i^\phi(s_i^{\mathrm{in},\phi}(t,x);t,x)=0, & m+1\le i\le n,\ (t,x)\in\mathcal{I}_i^\phi,\\
0, & 1\le i\le n,\ (t,x)\in\mathcal{J}_i^\phi.
\end{cases}$$

The existence of the entry and exit times $s_i^{\mathrm{in},\phi}(t,x)$ and $s_i^{\mathrm{out},\phi}(t,x)$ follows from Assumption 1 and $\|\phi\|_{C^1([0,\infty)\times[0,1])^n}<\infty$.

For $1\le i\le n$, let

$$\mathrm{Dom}(\chi_i^\phi)=\{(s,t,x)\in[0,\infty)\times[0,\infty)\times[0,1]:\ s\in[s_i^{\mathrm{in},\phi}(t,x),s_i^{\mathrm{out},\phi}(t,x)]\}.$$

For $1\le i\le n$ and for (s,t,x) in $\mathrm{Dom}(\chi_i^\phi)$, we have

$$\begin{aligned}
\partial_x\chi_i^\phi(s;t,x)=&\ \exp\left\{\int_t^s[\partial_x\lambda_i(\chi_i^\phi(\tau;t,x),\phi(\tau,\chi_i^\phi(\tau;t,x)))\right.\\
&\left.\qquad+\nabla_y\lambda_i(\chi_i^\phi(\tau;t,x),\phi(\tau,\chi_i^\phi(\tau;t,x)))\partial_x\phi(\tau,\chi_i^\phi(\tau;t,x))]\mathrm{d}\tau\right\},\\
\partial_t\chi_i^\phi(s;t,x)=&-\lambda_i(x,\phi(t,x))\partial_x\chi_i^\phi(s;t,x).
\end{aligned} \tag{16}$$

Differentiating (13), we obtain that for (t,x) in $[0,\infty)\times[0,1]$,

$$\partial_\nu s_i^{\text{out},\phi}(t,x)=\begin{cases} -\dfrac{\partial_\nu \chi_i^\phi(s_i^{\text{out},\phi}(t,x);t,x)}{\lambda_i(0,\phi(s_i^{\text{out},\phi}(t,x),0))}, & 1\le i\le m,\\ -\dfrac{\partial_\nu \chi_i^\phi(s_i^{\text{out},\phi}(t,x);t,x)}{\lambda_i(1,\phi(s_i^{\text{out},\phi}(t,x),1))}, & m+1\le i\le n,\end{cases} \tag{17}$$

with ∂_ν is ∂_t or ∂_x.

2.2 Properties for coefficients

Let us provide some properties of the coefficients in system (1)-(4). Denote by $\|\cdot\|$ the infinity norm for vector and matrix. For some constant $\delta>0$, denote $\mathcal{B}(\delta)^n=\{y\in\mathbb{R}^n:\|y\|\le\delta\}$. It follows from Assumption 2 and (9), and Inverse Function Theorem that there exist a neighborhood $\mathcal{U}\subset\mathbb{R}^m$ of the origin, a neighborhood $\mathcal{V}\subset\mathbb{R}^p$ of the origin and right inverse function $Q^\dagger:\mathcal{V}\to\mathcal{U}$ such that

$$Q^\dagger\in C^2(\mathcal{V})^m,\quad Q^\dagger(0)=0,\quad Q(Q^\dagger(a))=a,\ \forall a\in\mathcal{V}. \tag{18}$$

Note that $Q^\dagger$ is not necessarily unique. Take $\delta>\delta_0>0$ such that $\mathcal{B}(\delta_0)^p\subset\mathcal{V}$ and $\mathcal{U}\subset\mathcal{B}(\delta)^m$.

Let $\delta>\delta_0>0$, and let ϕ and φ in $C^1([0,\infty)\times[0,1])^n$ satisfy $\|\phi\|_{C^1([0,\infty)\times[0,1])^n}\le\delta$ and $\|\varphi\|_{C^1([0,\infty)\times[0,1])^n}\le\delta$. It follows from Assumptions 1 and 2 and (16) and (17) that there exist positive constants C_{δ_0}, C_δ and c_δ such that

1. for $t\ge0$, $0\le x\le1$ and $1\le i\le n$, we have

$$c_\delta\le|\lambda_i(x,\phi(t,x))|\le C_\delta,\quad |\partial_x\lambda_i(x,\phi(t,x))|\le C_\delta,\quad \|\nabla_y\lambda_i(x,\phi(t,x))\|\le C_\delta, \tag{19}$$

   and therefore,

$$t-s_i^{\text{in},\phi}(t,x)\le c_\delta^{-1},\quad s_i^{\text{out},\phi}(t,x)-t\le c_\delta^{-1}, \tag{20}$$

2. for $1\le i\le p$, $1\le j\le m$ and for a in $\mathcal{B}(\delta_0)^p$ and b in $\mathcal{B}(\delta)^m$, we have

$$\begin{aligned} &|Q_i(b)|\le C_\delta\|b\|,\quad \|\nabla Q_i(b)\|\le C_\delta,\quad \left\|\frac{\partial^2 Q_i}{\partial b^2}(b)\right\|\le C_\delta,\\ &|Q_j^\dagger(a)|\le C_{\delta_0}\|a\|,\quad \|\nabla Q_j^\dagger(a)\|\le C_{\delta_0},\quad \left\|\frac{\partial^2 Q_j^\dagger}{\partial a^2}(a)\right\|\le C_{\delta_0},\end{aligned} \tag{21}$$

   where $\frac{\partial^2 Q_i}{\partial b^2}$ and $\frac{\partial^2 Q_j^\dagger}{\partial a^2}$ are the Hessian matrices of Q_i and $Q_j^\dagger$, respectively;

3. for $1 \leq i \leq n$ and (s,t,x) in $\mathrm{Dom}(\chi_i^\phi)$, we have

$$\begin{aligned}
|\partial_x \chi_i^\phi(s;t,x)| &\leq \exp\left(\frac{C_\delta}{c_\delta}(1+\|\phi\|_{C^1([0,\infty)\times[0,1])^n})\right), \\
|\partial_t \chi_i^\phi(s;t,x)| &\leq C_\delta \exp\left(\frac{C_\delta}{c_\delta}(1+\|\phi\|_{C^1([0,\infty)\times[0,1])^n})\right);
\end{aligned} \tag{22}$$

4. for $t \geq 0$, $0 \leq x \leq 1$ and $1 \leq i \leq n$, we have

$$\begin{aligned}
|\partial_x s_i^{\mathrm{out},\phi}(t,x)| &\leq \frac{1}{c_\delta}\exp\left(\frac{C_\delta}{c_\delta}(1+\|\phi\|_{C^1([0,\infty)\times[0,1])^n})\right), \\
|\partial_t s_i^{\mathrm{out},\phi}(t,x)| &\leq \frac{C_\delta}{c_\delta}\exp\left(\frac{C_\delta}{c_\delta}(1+\|\phi\|_{C^1([0,\infty)\times[0,1])^n})\right);
\end{aligned} \tag{23}$$

5. for $1 \leq i \leq n$, and (s,t,x) in $\mathrm{Dom}(\chi_i^\phi) \cap \mathrm{Dom}(\chi_i^\varphi)$, we have

$$\begin{aligned}
&|\chi_i^\phi(s;t,x) - \chi_i^\varphi(s;t,x)| \\
&= \left| \int_t^s [\lambda_i(\chi_i^\phi(\sigma;t,x), \phi(\sigma, \chi_i^\phi(\sigma;t,x))) - \lambda_i(\chi_i^\varphi(\sigma;t,x), \varphi(\sigma, \chi_i^\varphi(\sigma;t,x)))]\mathrm{d}\sigma \right| \\
&\leq C_\delta \int_{\min\{s,t\}}^{\max\{s,t\}} [\|\phi-\varphi\|_{C^0([0,\infty)\times[0,1])^n} + (1+\delta)|\chi_i^\phi(\sigma;t,x) - \chi_i^\varphi(\sigma;t,x)|]\mathrm{d}\sigma,
\end{aligned}$$

and therefore, from (20) and Gronwall's inequality, we have

$$|\chi_i^\phi(s;t,x) - \chi_i^\varphi(s;t,x)| \leq \frac{C_\delta}{c_\delta}\exp\left(\frac{C_\delta(1+\delta)}{c_\delta}\right)\|\phi-\varphi\|_{C^0([0,\infty)\times[0,1])^n}; \tag{24}$$

6. for $t \geq 0$, $0 \leq x \leq 1$ and $1 \leq i \leq n$, we have

$$\begin{aligned}
&\int_t^{s_i^{\mathrm{out},\phi}(t,x)} \lambda_i(\chi_i^\phi(\sigma;t,x), \phi(\sigma, \chi_i^\phi(\sigma;t,x)))\mathrm{d}\sigma \\
&\qquad\qquad = \int_t^{s_i^{\mathrm{out},\varphi}(t,x)} \lambda_i(\chi_i^\varphi(\sigma;t,x), \varphi(\sigma, \chi_i^\varphi(\sigma;t,x)))]\mathrm{d}\sigma,
\end{aligned}$$

and therefore, from (24), we have

$$\begin{aligned}
&|s_i^{\mathrm{out},\phi}(t,x) - s_i^{\mathrm{out},\varphi}(t,x)| \\
&\qquad \leq \frac{C_\delta}{c_\delta^2}\left(1 + \frac{C_\delta(1+\delta)}{c_\delta}\exp\left(\frac{C_\delta(1+\delta)}{c_\delta}\right)\right)\|\phi-\varphi\|_{C^0([0,\infty)\times[0,1])^n}.
\end{aligned} \tag{25}$$

2.3 Properties for the modulus of continuity for continuous functions

Let $\mathcal{X}$ and $\mathcal{Y}$ be domains in $\mathbb{R}^{\tilde{m}}$ and $\mathbb{R}^{\tilde{n}}$, respectively. We introduce the modulus of continuity for continuous function $\phi : \mathcal{X} \to \mathcal{Y}$,

$$\rho(\alpha|\phi) = \sup_{\substack{\|x'-x''\|\leq\alpha, \\ x', x''\in\mathcal{X}}} \|\phi(x') - \phi(x'')\|, \quad \alpha \geq 0.$$

We provide some elementary facts for the modulus of continuity used in this paper.

1. For continuous functions ϕ_1, $\phi_2 : \mathcal{X} \to \mathbb{R}$, we have

$$\rho(\alpha|\phi_1\phi_2) \leq \|\phi_1\|_{C^0(\mathcal{X})}\rho(\alpha|\phi_2) + \|\phi_2\|_{C^0(\mathcal{X})}\rho(\alpha|\phi_1). \tag{26}$$

2. Let $\mathcal{Z}$ be domain in $\mathbb{R}^{\tilde{k}}$. For continuous functions $\phi_1 : \mathcal{X} \to \mathcal{Y}$ and $\phi_2 : \mathcal{Y} \to \mathcal{Z}$, we have

$$\rho(\alpha|\phi_2 \circ \phi_1) \leq \rho(\rho(\alpha|\phi_1)|\phi_2). \tag{27}$$

3. For positive constant C, there exists constant $C' > 0$ such that for continuous function $\phi : \mathcal{X} \to \mathcal{Y}$, we have

$$\rho(C\alpha|\phi) \leq C'\rho(\alpha|\phi). \tag{28}$$

3 Proof of Theorem 5

We prove Theorem 5 in three steps. In Section 3.1, we provide the design of the feedback regulator. In Section 3.2, we prove the well-posedness of system (1)-(4) with the feedback regulator. Finite-time output regulation is obtained in Section 3.3. In what follows, assume that the assumptions in Theorem 5 hold.

3.1 Design of the feedback regulator

Recalling the properties in Section 2.2, let us assume that $\|w^0\|_{C^1([0,1])^n} < \delta$ and $\|r\|_{C^1([0,\infty))^p} < \delta_0$. In order to achieve the finite-time output regulation, we divide the state w into two parts: one part possesses the finite-time stable properties, and the other part meets the output regulation objective. To this end, we split w into two new variables. For (t, x) in $[0, \infty) \times [0, 1]$, let

$$w(t, x) = z(t, x) + \Pi(t, x),$$

where $z = (z_1, \cdots, z_n)^\top : [0, \infty) \times [0, 1] \to \mathbb{R}^n$ and $\Pi = (\Pi_1, \cdots, \Pi_n)^\top : [0, \infty) \times [0, 1] \to \mathbb{R}^n$ are the new variables. We hope that the tracking error $e_y(t)$ defined in (7) depends only on z and z-system is finite-time stable. To this end, we determine the equations for z and Π through the following four steps.

1. It follows from (1) that for (t,x) in $[0,\infty)\times[0,1]$,

$$\begin{aligned}\partial_t z(t,x) + \Lambda(x, z(t,x) + \Pi(t,x))\partial_x z(t,x) \\ = -\partial_t \Pi(t,x) - \Lambda(x, z(t,x)+\Pi(t,x))\partial_x \Pi(t,x).\end{aligned} \tag{29}$$

   To ensure that the system of z is finite-time stable, we aim to eliminate any inhomogeneous terms in (29). Therefore, we set

$$\partial_t \Pi(t,x) + \Lambda(x, z(t,x)+\Pi(t,x))\partial_x \Pi(t,x) = 0, \quad t \geq 0,\ 0 \leq x \leq 1. \tag{30}$$

2. For the boundary condition at $x=0$, from (2), we obtain

$$z_+(t,0) = Q(z_-(t,0) + \Pi_-(t,0)) - \Pi_+(t,0), \quad t \geq 0.$$

   In order to establish the implication

$$(z_-(t,0)=0) \Rightarrow (z_+(t,0)=0), \quad t \geq 0,$$

   let

$$\Pi_+(t,0) = Q(\Pi_-(t,0)), \quad t \geq 0. \tag{31}$$

3. For $T > T_0$, set $\tau_0 = T - T_0$. For the boundary condition at $x=1$, from (3), we have

$$z_-(t,1) = w_-(t,1) - \Pi_-(t,1) = u(t) - \Pi_-(t,1), \quad t \geq 0.$$

   To satisfy the compatibility condition at $x=1$ and to establish

$$z_-(t,1) = 0, \quad t \geq \tau_0/2,$$

   we set

$$u(t) = \zeta(t) + (\mathrm{Id}_m - \eta(t))\Pi_-(t,1), \quad t \geq 0, \tag{32}$$

   where Id_m is $m\times m$ identity matrix, and $\zeta = (\zeta_1,\cdots,\zeta_m)^\top$ in $C^1([0,\infty))^m$ and $\eta = \mathrm{diag}(\eta_1,\cdots,\eta_m)$ in $C^1([0,\infty))^{m\times m}$ satisfy

$$\begin{aligned}&\zeta_i(0) = w_i^0(1), \quad \zeta_i'(0) = -\lambda_i(1, w^0(1))\partial_x w_i^0(1), \quad \eta_i(0) = 1, \quad \eta_i'(0) = 0, \\ &\zeta_i(t) = \eta_i(t) = 0 \quad \text{for } t \geq \min\{\tau_0/2, 1\},\end{aligned} \tag{33}$$

   and

$$\begin{aligned}&\|\zeta_i\|_{C^0([0,\infty))} \leq C_\delta \|w^0\|_{C^1([0,1])^n}, \quad \|\eta_i\|_{C^0([0,\infty))} \leq C, \\ &\|\zeta_i'\|_{C^0([0,\infty))} \leq C_\delta(1+\tau_0^{-1})\|w^0\|_{C^1([0,1])^n}, \quad \|\eta_i'\|_{C^0([0,\infty))} \leq C\tau_0^{-1},\end{aligned} \tag{34}$$

   for $1 \leq i \leq m$ and for some constants $C_\delta, C > 0$. We provide the precise dynamics for ζ_i and η_i and the proof of (34) in Appendix A.

4. It follows from (6) and (7) that

$$e_y(t) = w_+(t,1) - r(t) = z_+(t,1) + \Pi_+(t,1) - r(t), \quad t \geq 0.$$

In order to ensure the tracking error $e_y(t)$ depend only on z, we set

$$\Pi_+(t,1) = r(t), \quad t \geq 0. \tag{35}$$

The finite-time stability of z-system

$$\begin{aligned}
&\partial_t z(t,x) + \Lambda(x, z(t,x) + \Pi(t,x))\partial_x z(t,x) = 0, && t \geq 0,\ 0 \leq x \leq 1,\\
&z_+(t,0) = Q(z_-(t,0) + \Pi_-(t,0)) - \Pi_+(t,0), && t \geq 0,\\
&z_-(t,1) = \zeta(t) - \eta(t)\Pi_-(t,1), && t \geq 0,\\
&z(0,x) = w^0(x) - \Pi(0,x), && 0 \leq x \leq 1,
\end{aligned} \tag{36}$$

is discussed in Section 3.3. By (32), we can determine a feedback regulator $u(t)$ from (30), (31) and (35), which we refer to as the regulator equations in the output regulation problem.

Let $Q^\dagger$ be the right inverse function of Q. Notice that $w(t,x) = z(t,x) + \Pi(t,x)$. Integrating (30) along the characteristics $\chi_i^w(s;t,x)$, $1 \leq i \leq n$, and using the boundary conditions (31) and (35), we can choose $\Pi_-(t,1)$ as follows, for $t \geq 0$ and $1 \leq i \leq m$,

$$\Pi_i(t,1) = Q_i^\dagger(\tilde{r}_i^w(t)), \tag{37}$$

where

$$\tilde{r}_i^w(t) = \begin{pmatrix} r_1(s_{m+1}^{\text{out},w}(s_i^{\text{out},w}(t,1),0)) \\ \vdots \\ r_p(s_{m+p}^{\text{out},w}(s_i^{\text{out},w}(t,1),0)) \end{pmatrix}. \tag{38}$$

Remark 2 The choice of $\Pi_-(t,1)$ is not unique because the right inverse function $Q^\dagger$ is not necessarily unique.

Consequently, by considering the feedback regulator (32) with (37) for system (1)-(4), we obtain the system

$$\begin{aligned}
&\partial_t w(t,x) + \Lambda(x, w(t,x))\partial_x w(t,x) = 0, && t \geq 0,\ 0 \leq x \leq 1,\\
&w_+(t,0) = Q(w_-(t,0)), && t \geq 0,\\
&w_i(t,1) = \zeta_i(t) + (1 - \eta_i(t))Q_i^\dagger(\tilde{r}_i^w(t)), && t \geq 0,\ 1 \leq i \leq m,\\
&w(0,x) = w^0(x), && 0 \leq x \leq 1.
\end{aligned} \tag{39}$$

Remark 3 By the well-posedness result given in next subsection, the feedback regulator (32) with (37) is well-defined by the current state $w(t,\cdot)$ and the priori known reference signal r.

3.2 Well-posedness of (39)

In this section, the well-posedness for quasi-linear hyperbolic system (39) with non-linear, non-local boundary conditions. The proof is inspired by the methods used in [2, 22]. The main result of this section is the following lemma.

Lemma 6 *Let Λ, Q, w^0 and r satisfy Assumptions 1 to 4. Let T_0 be defined by (8) and (10). Let $\delta > \delta_0 > 0$ be given in Section 2.2. For any $T > T_0$, set $\tau_0 = T - T_0$. There exists $\varepsilon = \varepsilon(\delta, \delta_0, \tau_0)$ in $(0, \delta_0)$, such that if $\|w^0\|_{C^1([0,1])^n} < \varepsilon$ and $\|r\|_{C^1([0,\infty))^p} < \varepsilon$, then there exists a unique solution w in $C^1([0,\infty) \times [0,1])^n$ to system (39). Moreover, there exists a positive constant C, independent of w^0, r, τ_0 and ε, such that*

$$\|w\|_{C^1([0,\infty)\times[0,1])^n} \leq C(1 + \tau_0^{-1})(\|w^0\|_{C^1([0,1])^n} + \|r\|_{C^1([0,\infty))^p}). \tag{40}$$

Proof We use an iterative method to prove well-posedness of (39). We split the proof into four steps.

1. We show that the iterative sequence remains within the same local C^1 function subspace.
2. We prove that the iterative sequence converges in C^0 function space.
3. We use the Ascoli theorem to demonstrate that the sequence has a convergent subsequence in the C^1 function subspace.
4. We prove the uniqueness.

Step 1. Let $\|w^0\|_{C^1([0,1])^n} < \delta$ and $\|r\|_{C^1([0,\infty))^p} < \delta_0$. Let $\tau_0 = T - T_0$. For $1 \leq i \leq m$, let ζ_i and η_i be defined in Appendix A satisfying (33) and (34). It follows from Assumption 4 that there exist $T^* > 1 + 2c_\delta^{-1}$ and r^* in $\mathbb{R}^p$ satisfying $\|r^*\| \leq \|r\|_{C^1([0,\infty))^p}$ such that

$$r(t) = r^*, \quad t \geq T^*. \tag{41}$$

We set

$$\mathcal{D}(\theta) = \{\phi \in C^1([0,\infty) \times [0,1])^n : \|\phi\|_{C^1([0,\infty)\times[0,1])^n} < \theta, \\ \phi(0,\cdot) = w^0,\ \phi_+(t,\cdot) = r^* \text{ for } t \geq T^*,\ \phi_-(t,\cdot) = Q^\dagger(r^*) \text{ for } t \geq T^*\}.$$

In what follows, for notational ease, we ignore the dependence of constants on δ and δ_0. Fix an appropriate $w^{(0)}$ such that $w^{(0)}$ is in $\mathcal{D}(\delta)$ and $\|w^{(0)}\|_{C^1([0,\infty)\times[0,1])^n} \leq C_0(\|w^0\|_{C^1([0,1])^n} + \|r\|_{C^1([0,\infty))^p})$ for some constant $C_0 > 0$. For $l \geq 0$, let $w^{(l+1)}$ be the unique C^1-solution to

$$\begin{cases} \partial_t w^{(l+1)}(t,x) + \Lambda(x, w^{(l)}(t,x))\partial_x w^{(l+1)}(t,x) = 0, & t \geq 0,\ 0 \leq x \leq 1, \\ w_+^{(l+1)}(t,0) = Q(w_-^{(l+1)}(t,0)), & t \geq 0, \\ w_i^{(l+1)}(t,1) = \zeta_i(t) + (1 - \eta_i(t))Q_i^\dagger(\tilde{r}_i^{(l)}(t)), & t \geq 0,\ 1 \leq i \leq m, \\ w^{(l+1)}(0,x) = w^0(x), & 0 \leq x \leq 1, \end{cases} \tag{42}$$

where and in what follows, for simplicity, we denote $\chi_j^{(l)} = \chi_j^{w^{(l)}}$, $s_j^{\text{in},(l)} = s_j^{\text{in},w^{(l)}}$, $s_j^{\text{out},(l)} = s_j^{\text{out},w^{(l)}}$, $\mathcal{I}_j^{(l)} = \mathcal{I}_j^{w^{(l)}}$, $\mathcal{J}_j^{(l)} = \mathcal{J}_j^{w^{(l)}}$ and $\tilde{r}_i^{(l)}(t) = \tilde{r}_i^{w^{(l)}}(t)$, for $1 \leq j \leq n$, $1 \leq i \leq m$ and $l \geq 0$. Integrating (42) along the characteristics $\chi_i^{(l)}$, we obtain that for $1 \leq i \leq m$,

$$w_i^{(l+1)}(t,x) = \begin{cases} w_i^{(l+1)}(s_i^{\text{in},(l)}(t,x), 1), & (t,x) \in \mathcal{I}_i^{(l)}, \\ w_i^0(\chi_i^{(l)}(0;t,x)), & (t,x) \in \mathcal{J}_i^{(l)}, \end{cases} \tag{43}$$

and for $m+1 \le j \le n$,

$$w_j^{(l+1)}(t,x) = \begin{cases} Q_{j-m}(w_-^{(l+1)}(s_j^{\mathrm{in},(l)}(t,x),0)), & (t,x) \in \mathcal{I}_j^{(l)}, \\ w_j^0(\chi_j^{(l)}(0;t,x)), & (t,x) \in \mathcal{J}_j^{(l)}. \end{cases} \tag{44}$$

By (18), (20), (33), (41) and $T^* > 1 + 2c_\delta^{-1}$, for any $w^{(l)}$ in $\mathcal{D}(\delta)$, (t,x) in $[T^*,\infty)\times[0,1]$, $1 \le i \le m$ and $m+1 \le j \le n$, we have that

$$s_i^{\mathrm{in},(l)}(s_j^{\mathrm{in},(l)}(t,x),0) > 1,$$

and therefore, for $1 \le i \le m$ and $m+1 \le j \le n$,

$$w_i^{(l+1)}(t,x) = Q_i^\dagger(\tilde r_i^{(l)}(s_i^{\mathrm{in},(l)}(t,x))) = Q_i^\dagger \begin{pmatrix} r_1(s_{m+1}^{\mathrm{out},(l)}(s_i^{\mathrm{out},(l)}(t,x),0)) \\ \vdots \\ r_p(s_{m+p}^{\mathrm{out},(l)}(s_i^{\mathrm{out},(l)}(t,x),0)) \end{pmatrix} = Q_i^\dagger(r^*),$$

$$\begin{aligned} w_j^{(l+1)}(t,x) &= Q_{j-m}\begin{pmatrix} Q_1^\dagger(\tilde r_1^{(l)}(s_1^{\mathrm{in},(l)}(s_j^{\mathrm{in},(l)}(t,x),0))) \\ \vdots \\ Q_m^\dagger(\tilde r_m^{(l)}(s_m^{\mathrm{in},(l)}(s_j^{\mathrm{in},(l)}(t,x),0))) \end{pmatrix} \\ &= Q_{j-m}\circ Q^\dagger \begin{pmatrix} r_1(s_{m+1}^{\mathrm{out},(l)}(s_j^{\mathrm{in},(l)}(t,x),0)) \\ \vdots \\ r_p(s_{m+p}^{\mathrm{out},(l)}(s_j^{\mathrm{in},(l)}(t,x),0)) \end{pmatrix} \\ &= r_{j-m}(s_j^{\mathrm{out},(l)}(s_j^{\mathrm{in},(l)}(t,x),0)) = r_{j-m}(s_j^{\mathrm{out},(l)}(t,x)) = r_{j-m}^*. \end{aligned}$$

Let $W^{(l)}(t,x) = \partial_t w^{(l)}(t,x)$ for $t \ge 0$, $0 \le x \le 1$ and $l \ge 0$. We have

$$\begin{cases} \partial_t W^{(l+1)}(t,x) + \Lambda(x,w^{(l)}(t,x))\partial_x W^{(l+1)}(t,x) = A^{(l)}(t,x)W^{(l+1)}(t,x), & t\ge 0,\ 0\le x\le 1, \\ W_+^{(l+1)}(t,0) = \nabla Q(w_-^{(l+1)}(t,0))W_-^{(l+1)}(t,0), & t \ge 0, \\ W_i^{(l+1)}(t,1) = \zeta_i'(t) - \eta_i'(t)Q_i^\dagger(\tilde r_i^{(l)}(t)) + (1-\eta_i(t))\nabla Q_i^\dagger(\tilde r_i^{(l)}(t))(\tilde r_i^{(l)})'(t) & t\ge 0,\ 1\le i\le m, \\ W^{(l+1)}(0,x) = -\Lambda(x,w^0(x))\partial_x w^0(x), & 0 \le x \le 1, \end{cases} \tag{45}$$

where

$$(\tilde r_i^{(l)})'(t) = \begin{pmatrix} r_1'(s_{m+1}^{\mathrm{out},(l)}(s_i^{\mathrm{out},(l)}(t,1),0))\partial_t s_{m+1}^{\mathrm{out},(l)}(s_i^{\mathrm{out},(l)}(t,1),0) \\ \vdots \\ r_p'(s_{m+p}^{\mathrm{out},(l)}(s_i^{\mathrm{out},(l)}(t,1),0))\partial_t s_{m+p}^{\mathrm{out},(l)}(s_i^{\mathrm{out},(l)}(t,1),0) \end{pmatrix} \partial_t s_i^{\mathrm{out},(l)}(t,1), \tag{46}$$

and

$$\begin{aligned} A^{(l)}(t,x) &:= \mathrm{diag}(a_1^{(l)}(t,x),\dots,a_n^{(l)}(t,x)) \\ &= \mathrm{diag}(\nabla_y\lambda_1(x,w^{(l)}(t,x))W^{(l)}(t,x),\dots,\nabla_y\lambda_n(x,w^{(l)}(t,x))W^{(l)}(t,x))\Lambda^{-1}(x,w^{(l)}(t,x)). \end{aligned} \tag{47}$$

It follows from (21), (34), (38), (43) and (44) that there exists a constant $C_1 > 0$ such that for $w^{(l)}$ in $\mathcal{D}(\delta)$, $t \ge 0$ and $0 \le x \le 1$,

$$\|w_+^{(l+1)}(t,x)\| \le C_1(\|w^0\|_{C^0([0,1])^n} + \|w_-^{(l+1)}\|_{C^0([0,\infty)\times[0,1])^m}),$$

$$\|w_-^{(l+1)}(t,x)\| \le C_1(\|w^0\|_{C^1([0,1])^n} + \|r\|_{C^0([0,\infty))^p}),$$

Thus, there exists a constant $C_2 > 0$ such that

$$\|w^{(l+1)}\|_{C^0([0,\infty)\times[0,1])^n} \le C_2(\|w^0\|_{C^1([0,1])^n} + \|r\|_{C^0([0,\infty))^p}). \tag{48}$$

Integrating (45) along the characteristics $\chi_i^{(l)}$ and using (19)-(21), (23), (34), (38) and (46), we obtain that there exists a constant $C_3 > 0$ such that for $w^{(l)}$ in $\mathcal{D}(\delta)$, $t \ge 0$ and $0 \le x \le 1$,

$$\begin{aligned}
\|W_+^{(l+1)}(t,x)\| \le{}& C_3(\|w^0\|_{C^1([0,1])^n} + \|W_-^{(l+1)}\|_{C^0([0,\infty)\times[0,1])^m} \\
&+ \|w^{(l)}\|_{C^1([0,\infty)\times[0,1])^n}\|W_+^{(l+1)}\|_{C^0([0,\infty)\times[0,1])^p}), \\
\|W_-^{(l+1)}(t,x)\| \le{}& C_3[(1+\tau_0^{-1})(\|w^0\|_{C^1([0,1])^n} + \|r\|_{C^1([0,\infty))^p}) \\
&+ \|w^{(l)}\|_{C^1([0,\infty)\times[0,1])^n}\|W_-^{(l+1)}\|_{C^0([0,\infty)\times[0,1])^m}].
\end{aligned}$$

Let ε' in $(0,\delta)$ be sufficiently small (independent of τ_0) and assume that

$$\|w^{(l)}\|_{C^1([0,\infty)\times[0,1])^n} < \varepsilon'.$$

Then we have for some constant $C_4 > 0$,

$$\|W^{(l+1)}\|_{C^0([0,\infty)\times[0,1])^n} \le C_4(1+\tau_0^{-1})(\|w^0\|_{C^1([0,1])^n} + \|r\|_{C^1([0,\infty))^p}). \tag{49}$$

It follows from (19), (42), (48) and (49) that there exists a constant $C_5 > 0$ such that if $w^{(l)}$ is in $\mathcal{D}(\varepsilon')$,

$$\|w^{(l+1)}\|_{C^1([0,\infty)\times[0,1])^n} \le C_5(1+\tau_0^{-1})(\|w^0\|_{C^1([0,1])^n} + \|r\|_{C^1([0,\infty))^p}).$$

Thus, there exists a constant $\varepsilon = \varepsilon(\varepsilon', \tau_0)$ in $(0, \varepsilon')$ independent of w^0 and r such that

$$\|w^{(l+1)}\|_{C^1([0,\infty)\times[0,1])^n} < \varepsilon',$$

if

$$\|w^0\|_{C^1([0,1])^n} \le \varepsilon, \quad \|r\|_{C^1([0,\infty))^p} \le \varepsilon. \tag{50}$$

Consequently, for ε' in $(0,\delta)$ small enough, assume that w^0 and r satisfy (50). Choose an appropriate $w^{(0)}$ such that $w^{(0)}$ is in $\mathcal{D}(\varepsilon')$ and $\|w^{(0)}\|_{C^1([0,\infty)\times[0,1])^n} \le C_5(1+\tau_0^{-1})(\|w^0\|_{C^1([0,1])^n} + \|r\|_{C^1([0,\infty))^p})$. From (42), we get a sequence $\{w^{(l)}\}_{l=0}^{\infty}$ in $\mathcal{D}(\varepsilon')$ such that

$$\|w^{(l)}\|_{C^1([0,\infty)\times[0,1])^n} \le C_5(1+\tau_0^{-1})(\|w^0\|_{C^1([0,1])^n} + \|r\|_{C^1([0,\infty))^p}), \quad l \ge 0. \tag{51}$$

Step 2. Let $V^{(l)}(t,x) = w^{(l)}(t,x) - w^{(l-1)}(t,x)$ for $t \ge 0$, $0 \le x \le 1$ and $l \ge 1$. We have

$$\begin{aligned}
&\partial_t V^{(l+1)}(t,x) + \Lambda(x, w^{(l)}(t,x))\partial_x V^{(l+1)}(t,x) + B^{(l)}(t,x) = 0, && t \ge 0,\ 0 \le x \le 1, \\
&V_+^{(l+1)}(t,0) = Q(w_-^{(l+1)}(t,0)) - Q(w_-^{(l)}(t,0)), && t \ge 0, \\
&V_i^{(l+1)}(t,1) = (1-\eta_i(t))\left[Q_i^\dagger(\tilde{r}_i^{(l)}(t)) - Q_i^\dagger(\tilde{r}_i^{(l-1)}(t))\right], && t \ge 0,\ 1 \le i \le m, \\
&V^{(l+1)}(0,x) = 0, && 0 \le x \le 1,
\end{aligned} \tag{52}$$

where

$$B^{(l)}(t,x) = [\Lambda(x, w^{(l)}(t,x)) - \Lambda(x, w^{(l-1)}(t,x))]\partial_x w^{(l)}(t,x).$$

Noticing that $\|w^{(l)}\|_{C^1([0,\infty)\times[0,1])^n} < \varepsilon'$, $l \geq 0$, and (50), it follows from (19), (21), (23), (25), (34) and (38) that there exists a constant $C_6 > 0$ such that for $t \geq 0$, $0 \leq x \leq 1$ and $1 \leq i \leq m$,

$$\begin{aligned}
&\|B^{(l)}(t,x)\| \leq C_6 \|w^{(l)}\|_{C^1([0,\infty)\times[0,1])^n} \|V^{(l)}(t,x)\| \leq C_6 \varepsilon' \|V^{(l)}(t,x)\|,\\
&\|Q(w_-^{(l+1)}(t,0)) - Q(w_-^{(l)}(t,0))\| \leq C_6 \|V_-^{(l+1)}(t,x)\|,\\
&\left|(1-\eta_i(t))\left[Q_i^\dagger(\tilde{r}_i^{(l)}(t)) - Q_i^\dagger(\tilde{r}_i^{(l-1)}(t))\right]\right|\\
&\leq C_6 \|r\|_{C^1([0,\infty))^p} \|V^{(l)}\|_{C^0([0,\infty)\times[0,1])^n} \leq C_6 \varepsilon' \|V^{(l)}\|_{C^0([0,\infty)\times[0,1])^n}.
\end{aligned} \tag{53}$$

Integrating (52) along the characteristics $\chi_i^{(l)}$ and using the estimates (53), we obtain that there exists a constant $C_7 > 0$ such that for $t \geq 0$ and $0 \leq x \leq 1$,

$$\begin{aligned}
&\|V_+^{(l+1)}(t,x)\| \leq C_7(\|V_-^{(l+1)}\|_{C^0([0,\infty)\times[0,1])^m} + \varepsilon' \|V^{(l)}\|_{C^0([0,\infty)\times[0,1])^n}),\\
&\|V_-^{(l+1)}(t,x)\| \leq C_7 \varepsilon' \|V^{(l)}\|_{C^0([0,\infty)\times[0,1])^n}.
\end{aligned}$$

It follows that

$$\|V^{(l+1)}\|_{C^0([0,\infty)\times[0,1])^n} \leq \frac{1}{2} \|V^{(l)}\|_{C^0([0,\infty)\times[0,1])^n},$$

if ε' is small enough. This implies $\{w^{(l)}\}_{l=0}^\infty$ converges in $C^0([0,\infty)\times[0,1])^n$.

Step 3. Noticing that for function ϕ in $\mathcal{D}(\varepsilon')$,

$$\|\phi\|_{C^1([0,\infty)\times[0,1])^n} = \|\phi\|_{C^1([0,T^*]\times[0,1])^n}.$$

Therefore, we only need to prove $\{w^{(l)}\}_{l=0}^\infty$ process a subsequence that converges in $C^1([0,T^*]\times[0,1])^n$. To this end, by Ascoli theorem, we need to prove $\{\partial_t w^{(l)}\}_{l=0}^\infty$ and $\{\partial_x w^{(l)}\}_{l=0}^\infty$ are uniformly equicontinuous on the domain $[0,T^*]\times[0,1]$. Recalling the notions of the modulus of continuity introduced in Section 2.3, it is sufficient to prove

$$\begin{aligned}
\rho(\alpha|\partial_t w^{(l)}) \to 0, \quad \text{as } \alpha \to 0,\\
\rho(\alpha|\partial_x w^{(l)}) \to 0, \quad \text{as } \alpha \to 0,
\end{aligned}$$

for all $l \geq 0$. Noticing that $\partial_x w^{(l)}(t,x) = -\Lambda(x, w^{(l-1)}(t,x))^{-1} \partial_t w^{(l)}(t,x)$, it follows from (19) that there exists a constant $C_8 > 0$ such that

$$\rho(\alpha|\partial_x w^{(l)}) \leq C_8(\rho(\alpha|\partial_t w^{(l)}) + \alpha).$$

Let us now estimate $\rho(\alpha|\partial_t w^{(l)})$, namely $\rho(\alpha|W^{(l)})$. For function ϕ defined on $[0,\infty)\times[0,1]$, denote

$$\rho(\alpha,\tau|\phi) = \sup_{\substack{|t'-t''|\leq\alpha, |x'-x''|\leq\alpha,\\ t',t''\in[0,\tau],\, x',x''\in[0,1]}} \|\phi(t',x') - \phi(t'',x'')\|, \quad \tau \geq 0,\ \alpha \geq 0.$$

Since $w^{(l)}$ is in $\mathcal{D}(\varepsilon')$, we have

$$\rho(\alpha|W^{(l)}) = \rho(\alpha, T^*|W^{(l)}) = \sup_{0\leq\tau\leq T^*} \rho(\alpha,\tau|W^{(l)}).$$

Let us estimate $\rho(\alpha,\tau|W^{(l)})$ for τ in $[0,T^*]$. Integrating (45) along the characteristics $\chi_i^{(l)}$ and recalling (14) and (15), we obtain

$$W_i^{(l+1)}(t,x) = I_i^{(l+1)}(s_i^{\mathrm{in},(l)}(t,x), \chi_i^{(l)}(0;t,x)) + J_i^{(l+1)}(t,x),\ (t,x) \in [0,\infty)\times[0,1],\ 1 \leq i \leq n,$$

where

$$J_i^{(l+1)}(t,x)=\int_{s_i^{\mathrm{in},(l)}(t,x)}^{t} a_i^{(l)}(s,\chi_i^{(l)}(s;t,x))W_i^{(l+1)}(s,\chi_i^{(l)}(s;t,x))\mathrm{d}s,\quad 1\le i\le n,$$

$$I_i^{(l+1)}(s_i^{\mathrm{in},(l)}(t,x),\chi_i^{(l)}(0;t,x))=\begin{cases} W_i^{(l+1)}(s_i^{\mathrm{in},(l)}(t,x),1), & 1\le i\le m,\ (t,x)\in\mathcal{I}_i^{(l)},\\ W_i^{(l+1)}(s_i^{\mathrm{in},(l)}(t,x),0), & m+1\le i\le n,\ (t,x)\in\mathcal{I}_i^{(l)},\\ W_i^{(l+1)}(0,\chi_i^{(l)}(0;t,x)), & 1\le i\le n,\ (t,x)\in\mathcal{J}_i^{(l)}.\end{cases}$$

Then we split the estimate of $\rho(\alpha,\tau|W^{(l+1)})$ into four parts.

1. Estimate of $\rho(\alpha,\tau|W_i^{(l+1)}(0,\chi_i^{(l)}(0;\cdot,\cdot)))$ for $1\le i\le n$. Due to (22), there exists a constant $C_9>0$ such that for $1\le i\le n$ and $(t',x'),(t'',x'')$ in $[0,\infty)\times[0,1]$,

   $$|\chi_i^{(l)}(0;t',x')-\chi_i^{(l)}(0;t'',x'')|\le\frac{C_9}{2}(|t'-t''|+|x'-x''|).$$

   Therefore, for $1\le i\le n$, we have the implication

   $$(|t'-t''|\le\alpha,\ |x'-x''|\le\alpha)\Rightarrow(|\chi_i^{(l)}(0;t',x')-\chi_i^{(l)}(0;t'',x'')|\le C_9\alpha).$$

   It follows from (28) that there exists a constant $C_{10}>0$ such that for $1\le i\le n$,

   $$\rho(\alpha,\tau|W_i^{(l+1)}(0,\chi_i^{(l)}(0;\cdot,\cdot)))\le C_{10}\rho(\alpha|W_i^{(l+1)}(0,\cdot)).$$

   Using (19), (26) and the expression of $W_i^{(l+1)}(0,\cdot)$ in (45), we obtain that there exists a constant $C_{11}>0$ such that for $1\le i\le n$,

   $$\rho(\alpha,\tau|W_i^{(l+1)}(0,\chi_j^{(l)}(0;\cdot,\cdot)))\le C_{11}(\rho(\alpha|\partial_x w^0)+\alpha).\tag{54}$$

2. Estimate of $\rho(\alpha,\tau|W_i^{(l+1)}(s_i^{\mathrm{in},(l)}(\cdot,\cdot),0))$ for $m+1\le i\le n$. Similarly, by (23) and (28), there exists a constant $C_{12}>0$ such that for $m+1\le i\le n$,

   $$\rho(\alpha,\tau|W_i^{(l+1)}(s_i^{\mathrm{in},(l)}(\cdot,\cdot),0))\le C_{12}\rho(\alpha,\tau|W_i^{(l+1)}(\cdot,0)).$$

   It follows from (21), (26), (27) and the expression of $W_+^{(l+1)}(t,0)$ in (45) that there exists a constant $C_{13}>0$ such that for $m+1\le i\le n$,

   $$\begin{aligned}\rho(\alpha,\tau|W_i^{(l+1)}(s_i^{\mathrm{in},(l)}(\cdot,\cdot),0))&\le C_{13}(\rho(\alpha,\tau|W_-^{(l+1)}(\cdot,0))+\alpha)\\&\le C_{13}(\rho(\alpha,\tau|W_-^{(l+1)})+\alpha).\end{aligned}\tag{55}$$

3. Estimate of $\rho(\alpha,\tau|W_i^{(l+1)}(s_i^{\mathrm{in},(l)}(\cdot,\cdot),1))$ for $1\le i\le m$. Similarly, by (23) and (28), there exists a constant $C_{14}>0$ such that for $1\le i\le m$,

   $$\rho(\alpha,\tau|W_i^{(l+1)}(s_i^{\mathrm{in},(l)}(\cdot,\cdot),1))\le C_{14}\rho(\alpha,\tau|W_i^{(l+1)}(\cdot,1)).$$

   Using (21), (23), (26)-(28), (34), (38), (46) and the expression of $W_-^{(l+1)}(\cdot,1)$ in (45), we obtain that there exist constants $C_{15}>0$ such that for $1\le i\le m$,

   $$\begin{aligned}\rho(\alpha,\tau|W_i^{(l+1)}(s_i^{\mathrm{in},(l)}(\cdot,\cdot),1))\le{}& C_{15}(\rho(\alpha|\zeta_i')+\rho(\alpha|\eta_i')+\rho(\alpha|r')\\&+\varepsilon'\max_{m+1\le j\le n}\rho(\alpha,\tau|\partial_t s_j^{\mathrm{out},(l)}(\cdot,0))+\varepsilon'\rho(\alpha,\tau|\partial_t s_i^{\mathrm{out},(l)}(\cdot,1))+(1+\tau_0^{-1})\alpha).\end{aligned}$$

   Since ζ_i' and η_i' are uniformly continuous on the domain $[0,\tau_0/2]$ for $1\le i\le m$, there exists a function $\rho_{\tau_0}:[0,\infty)\to[0,\infty)$, satisfying

   $$\rho_{\tau_0}(\alpha)\to0,\quad\text{as }\alpha\to0,$$

such that for $1 \le i \le m$,

$$\rho(\alpha, \tau | W_i^{(l+1)}(s_i^{\mathrm{in},(l)}(\cdot,\cdot),1)) \le C_{15}(\rho(\alpha|r') + \rho_{\tau_0}(\alpha) + \varepsilon' \max_{m+1\le j\le n} \rho(\alpha,\tau|\partial_t s_j^{\mathrm{out},(l)}(\cdot,0)) + \varepsilon' \rho(\alpha,\tau|\partial_t s_i^{\mathrm{out},(l)}(\cdot,1))).$$

Let us now estimate $\rho(\alpha,\tau|\partial_t s_j^{\mathrm{out},(l)}(\cdot,0))$ for $m+1 \le j \le n$. Similar estimate holds also for $\rho(\alpha,\tau|\partial_t s_i^{\mathrm{out},(l)}(\cdot,1))$ with $1 \le i \le m$. Noticing the expression of $\partial_t s_i^{\mathrm{out},(l)}(t,x)$ in (16) and (17), for (t,x) in $[0,\infty)\times[0,1]$ and $1 \le i \le n$, denote

$$F_i^{(l)}(t,x) = \partial_x \lambda_i(x, w^{(l)}(t,x)) + \nabla_y \lambda_i(x, w^{(l)}(t,x)) \partial_x w^{(l)}(t,x).$$

It follows from (19), (20), (22), (23) and (26)-(28) that there exist constants C_{16}, C_{17}, $C_{18} > 0$ such that for $m+1 \le j \le n$,

$$\begin{aligned}\rho(\alpha,\tau|\partial_t s_j^{\mathrm{out},(l)}(\cdot,0)) &\le C_{16}\left(\rho\left(\alpha,\tau\left|\int_{\cdot}^{s_j^{\mathrm{out},(l)}(\cdot,0)} F_j^{(l)}(\tau,\chi_j^{(l)}(\tau;\cdot,0))\mathrm{d}\tau\right.\right)+\alpha\right)\\ &\le C_{17}(\rho(\alpha,T^*|F_j^{(l)})+\alpha) \le C_{18}(\rho(\alpha,T^*|W^{(l)})+\alpha).\end{aligned}$$

Therefore, it follows that there exists a constant $C_{19} > 0$ such that for $1 \le i \le m$,

$$\rho(\alpha,\tau|W_i^{(l+1)}(s_i^{\mathrm{in},(l)}(\cdot,\cdot),1)) \le C_{19}(\rho(\alpha|r') + \rho_{\tau_0}(\alpha) + \varepsilon'\rho(\alpha,T^*|W^{(l)})). \tag{56}$$

4. Estimate of $\rho(\alpha,\tau|J_i^{(l+1)})$ for $1 \le i \le n$. It follows from (19), (22), (23) and (28), there exist constants C_{20}, $C_{21} > 0$ such that for $1 \le i \le n$,

$$\begin{aligned}\rho(\alpha,\tau|J_i^{(l+1)}) &\le C_{20}\left(\int_0^\tau \rho(\alpha|(a_i^{(l)}W_i^{(l+1)})(s,\chi_i^{(l)}(s;\cdot,\cdot)))\mathrm{d}s+\alpha\right)\\ &\le C_{21}\left(\int_0^\tau \rho(\alpha|(a_i^{(l)}W_i^{(l+1)})(s,\cdot))\mathrm{d}s+\alpha\right)\\ &\le C_{21}\left(\int_0^\tau \rho(\alpha,s|a_i^{(l)}W_i^{(l+1)})\mathrm{d}s+\alpha\right).\end{aligned}\tag{57}$$

Recalling (19), (26) and (47), there exists a constant $C_{22} > 0$ such that

$$\rho(\alpha,s|a_i^{(l)}W_i^{(l+1)}) \le C_{22}(\rho(\alpha,s|W^{(l)}) + \rho(\alpha,s|W_i^{(l+1)}) + \alpha). \tag{58}$$

Therefore, it follows from (57) and (58) that there exists a constant $C_{23} > 0$ such that

$$\rho(\alpha,\tau|J_i^{(l+1)}) \le C_{23}\left(\int_0^\tau [\rho(\alpha,s|W^{(l)}) + \rho(\alpha,s|W_i^{(l+1)})]\mathrm{d}s + (1+T^*)\alpha\right). \tag{59}$$

Consequently, it follows from (54), (55), (56) and (59) that there exists a constant $C_{24} > 0$ such that for τ in $[0,T^*]$,

$$\begin{aligned}&\rho(\alpha,\tau|W^{(l+1)})\\ &\le C_{24}\left(\int_0^\tau [\rho(\alpha,s|W^{(l)}) + \rho(\alpha,s|W^{(l+1)})]\mathrm{d}s + \hat{\rho}_{\tau_0,T^*}(\alpha,w^0,r') + \varepsilon'\rho(\alpha,T^*|W^{(l)})\right),\end{aligned}\tag{60}$$

where

$$\hat{\rho}_{\tau_0,T^*}(\alpha,w^0,r') = \rho(\alpha|\partial_x w^0) + \rho(\alpha|r') + \rho_{\tau_0}(\alpha) + T^*\alpha.$$

Without loss of generality, we assume that $C_{24} > 1$ and

$$\rho(\alpha,T^*|W^{(0)}) \le \hat{\rho}_{\tau_0,T^*}(\alpha,w^0,r'), \tag{61}$$

since $W^{(0)}$ is uniformly continuous on the domain $[0, T^*] \times [0, 1]$. We claim that for all $l \geq 0$ and for some large positive constant L,

$$\max_{0 \leq \tau \leq T^*} \rho(\alpha, \tau | W^{(l)}) \mathrm{e}^{-L\tau} \leq 3C_{24} \hat{\rho}_{\tau_0, T^*}(\alpha, w^0, r'), \tag{62}$$

provided that ε' is sufficiently small. Due to (61), the claim (62) holds for $l = 0$. Assume that (62) holds for some $l > 0$. It follows from (60) that for some $L > 0$,

$$\begin{aligned} &\rho(\alpha, \tau | W^{(l+1)}) \\ &\leq C_{24} \hat{\rho}_{\tau_0, T^*}(\alpha, w^0, r') \left(1 + \frac{3C_{24}}{L}(\mathrm{e}^{L\tau} - 1) + 3\varepsilon' C_{24} \mathrm{e}^{LT^*}\right) + C_{24} \int_0^\tau \rho(\alpha, s | W^{(l+1)}) \mathrm{d}s. \end{aligned}$$

By Gronwall's inequality, we obtain that for some $L > 0$,

$$\rho(\alpha, \tau | W^{(l+1)}) \leq C_{24} \hat{\rho}_{\tau_0, T^*}(\alpha, w^0, r') \left(1 + \frac{3C_{24}}{L}(\mathrm{e}^{L\tau} - 1) + 3\varepsilon' C_{24} \mathrm{e}^{LT^*}\right) \mathrm{e}^{C_{24}\tau},$$

which implies that for some $L > 0$,

$$\begin{aligned} &\rho(\alpha, \tau | W^{(l+1)}) \mathrm{e}^{-L\tau} \\ &\qquad \leq C_{24} \hat{\rho}_{\tau_0, T^*}(\alpha, w^0, r') \left(\frac{3C_{24}}{L} \mathrm{e}^{C_{24}\tau} + \mathrm{e}^{-(L - C_{24})\tau} \left(1 - \frac{3C_{24}}{L} + 3\varepsilon' C_{24} \mathrm{e}^{LT^*}\right)\right). \end{aligned}$$

It follows that

$$\rho(\alpha, \tau | W^{(l+1)}) \mathrm{e}^{-L\tau} \leq C_{24} \hat{\rho}_{\tau_0, T^*}(\alpha, w^0, r') \left(\frac{3C_{24}}{L} \mathrm{e}^{C_{24}T^*} + 1 - \frac{3C_{24}}{L} + 3\varepsilon' C_{24} \mathrm{e}^{LT^*}\right),$$

provided that

$$L > 3C_{24}. \tag{63}$$

Taking L large enough such that

$$\frac{3C_{24}}{L} \mathrm{e}^{C_{24}T^*} + 1 - \frac{3C_{24}}{L} \leq \frac{3}{2},$$

namely

$$L \geq 6C_{24}(\mathrm{e}^{C_{24}T^*} - 1), \tag{64}$$

and taking ε' small enough such that

$$0 < \varepsilon' \leq \frac{\mathrm{e}^{-LT^*}}{2C_{24}}, \tag{65}$$

we obtain that (62) holds for $l + 1$.

Consequently, for all $l \geq 0$ and for some large positive constant L satisfying (63) and (64), we have

$$\rho(\alpha | W^{(l)}) = \rho(\alpha, T^* | W^{(l)}) \leq 3C_{24} \mathrm{e}^{LT^*} \hat{\rho}_{\tau_0, T^*}(\alpha, w^0, r'),$$

provided that ε is small enough such that (65) holds. Since $\partial_x w^0$ and r' are uniformly continuous on the domain $[0, 1]$ and $[0, T^*]$ respectively, we have that $\{\partial_t w^{(l)}\}_{l=0}^\infty$ and $\{\partial_x w^{(l)}\}_{l=0}^\infty$ are uniformly equicontinuous on the domain $[0, T^*] \times [0, 1]$. Then by Ascoli theorem, we have that $\{w^{(l)}\}_{l=0}^\infty$ process a subsequence that converges in $C^1([0, T^*] \times [0, 1])^n$. It is clear that the limit is a C^1-solution to system (39).

Step 4. We next establish the uniqueness. Let w and $\hat{w}$ in $C^1([0,\infty)\times[0,1])^n$ be two solutions to system (39). Set $v = \hat{w} - w$ in $[0,\infty)\times[0,1]$. Then we have

$$\begin{aligned} &\partial_t v(t,x) + \Lambda(x, w(t,x))\partial_x v(t,x) + f(t,x,v(t,x)) = 0, && t \geq 0,\ 0 \leq x \leq 1,\\ &v_+(t,0) = Q(w_-(t,0) + v_-(t,0)) - Q(w_-(t,0)), && t \geq 0,\\ &v_i(t,1) = (1-\eta_i(t))\left[Q_i^\dagger(\tilde{r}_i^{w+v}(t)) - Q_i^\dagger(\tilde{r}_i^{w}(t))\right], && t \geq 0,\ 1 \leq i \leq m,\\ &v(0,x) = 0, && 0 \leq x \leq 1, \end{aligned} \tag{66}$$

where

$$f(t,x,v(t,x)) = [\Lambda(x, w(t,x) + v(t,x)) - \Lambda(x, w(t,x))]\partial_x \hat{w}(t,x).$$

Noticing $\|w\|_{C^1([0,\infty)\times[0,1])^n} < \varepsilon'$, $\|\hat{w}\|_{C^1([0,\infty)\times[0,1])^n} < \varepsilon'$, and (50), it follows from (19), (21) and (25) that there exists a constant $C_{25} > 0$ such that for $t \geq 0$, $0 \leq x \leq 1$ and $1 \leq i \leq m$,

$$\begin{aligned} |f(t,x,v(t,x))| &\leq C_{25}\|\hat{w}\|_{C^1([0,\infty)\times[0,1])^n}\|v(t,x)\| \leq C_{25}\varepsilon'\|v(t,x)\|,\\ \|Q(w_-(t,0) + v_-(t,0)) - Q(w_-(t,0))\| &\leq C_{25}\|v_-(t,x)\|,\\ \left|(1-\eta_i(t))\left[Q_i^\dagger(\tilde{r}_i^{w+v}(t)) - Q_i^\dagger(\tilde{r}_i^{w}(t))\right]\right| &\leq C_{25}\|r\|_{C^1([0,\infty))^p}\|v\|_{C^0([0,\infty)\times[0,1])^n}\\ &\leq C_{25}\varepsilon'\|v\|_{C^0([0,\infty)\times[0,1])^n}. \end{aligned} \tag{67}$$

Integrating (66) along the characteristics χ_i^w and using the estimates (67), we obtain that there exists a constant $C_{26} > 0$ such that for $t \geq 0$ and $0 \leq x \leq 1$,

$$\begin{aligned} &\|v_+(t,x)\| \leq C_{26}(\|v_-\|_{C^0([0,\infty)\times[0,1])^m} + \varepsilon'\|v\|_{C^0([0,\infty)\times[0,1])^n}),\\ &\|v_-(t,x)\| \leq C_{26}\varepsilon'\|v\|_{C^0([0,\infty)\times[0,1])^n}. \end{aligned}$$

It follows that

$$v = 0,$$

if ε' is small enough. Then the uniqueness follows. The estimate (40) follows from (51). The proof is complete. □

3.3 Finite-time stability of z-system

Let w in $C^1([0,\infty)\times[0,1])^n$ be the solution to system (39). Then we can write z-system (36) as

$$\begin{aligned} &\partial_t z(t,x) + \Lambda(x, w(t,x))\partial_x z(t,x) = 0, && t \geq 0,\ 0 \leq x \leq 1,\\ &z_+(t,0) = Q(z_-(t,0) + \Pi_-(t,0)) - \Pi_+(t,0), && t \geq 0,\\ &z_-(t,1) = \zeta(t) - \eta(t)\Pi_i(t,1), && t \geq 0,\\ &z(0,x) = w^0(x) - \Pi(0,x), && 0 \leq x \leq 1, \end{aligned} \tag{68}$$

where Π in $C^1([0,\infty)\times[0,1])^n$ is the solution to the regulator equations

$$\begin{aligned} &\partial_t \Pi(t,x) + \Lambda(x, w(t,x))\partial_x \Pi(t,x) = 0, && t \geq 0,\ 0 \leq x \leq 1,\\ &\Pi_+(t,0) = Q(\Pi_-(t,0)), && t \geq 0,\\ &\Pi_+(t,1) = r(t), && t \geq 0. \end{aligned} \tag{69}$$

From (33), we have that for $t \geq \tau_0/2$,

$$z_-(t, 1) = 0.$$

It follows from the characteristic method that

$$z_-(t, \cdot) = 0, \quad t \geq s_m^{\text{out},w}(\tau_0/2, 1).$$

Then by the boundary condition $\Pi_+(t, 0) = Q(\Pi_-(t, 0))$ in (69) and boundary condition $z_+(t, 0) = Q(z_-(t, 0) + \Pi_-(t, 0)) - \Pi_+(t, 0)$ in (68), we have

$$z_+(t, 0) = 0, \quad t \geq s_m^{\text{out},w}(\tau_0/2, 1).$$

Using the characteristic method again, we have

$$z_+(t, \cdot) = 0, \quad t \geq s_{m+1}^{\text{out},w}(s_m^{\text{out},w}(\tau_0/2, 1), 0).$$

Recalling τ_m and τ_{m+1} defined in (8), it follows from $\|w\|_{C^1([0,\infty)\times[0,1])^n} < \varepsilon'$ and (25) that for $t \geq 0$,

$$|s_m^{\text{out},w}(t, 1) - t - \tau_m| \leq \tau_0/4, \quad |s_{m+1}^{\text{out},w}(t, 0) - t - \tau_{m+1}| \leq \tau_0/4,$$

if $\varepsilon' \leq C_{27}\tau_0$ for some positive constant C_{27}. Then it follows from $\tau_0 = T - \tau_m - \tau_{m+1}$ that

$$s_{m+1}^{\text{out},w}(s_m^{\text{out},w}(\tau_0/2, 1), 0) \leq T.$$

Therefore, we have

$$z_+(t, \cdot) = 0, \quad t \geq T,$$

which implies that

$$e_y(t) = z_+(t, 1) = 0, \quad t \geq T.$$

This concludes the proof of Theorem 5.

4 Conclusion

This paper considers the finite-time output regulation problem for quasilinear hyperbolic systems. Under the assumption that the reference signal becomes a constant after some moment, we solve the single-boundary output regulation problem with single-boundary control. The rank condition at the uncontrolled boundary ensures that the control can propagate through this boundary condition to all state components, thereby achieving output regulation. We employ time-varying feedback regulator to achieve this goal. The output regulation problem for quasilinear hyperbolic systems still presents many open challenges. For instance, it would be worthwhile to explore whether the approaches developed in [25] can be extended to quasilinear hyperbolic systems. Specifically, one can examine the feasibility of achieving output regulation for such systems by a nonlinear time-invariant feedback regulator. Additionally, robust output regulation with respect to system parameters remains an unresolved issue.

Appendix A Dynamics for ζ and η

Here we provide the dynamics for ζ_i and η_i such that ζ_i and η_i satisfy (33) and (34) for $1 \le i \le m$. The dynamics are established by [2]. We restate them here and provide a more precise estimate with respect to τ_0. To simplify the notations we drop the subscript i in this appendix. We write $\zeta = \omega + \psi$ where ω and ψ satisfy the dynamics

$$\omega'(t) = -\frac{\alpha\omega(t)}{(\omega(t)^2 + \psi(t)^2)^{1/3}}, \quad \psi'(t) = -\frac{\beta\psi(t)}{(\omega(t)^2 + \psi(t)^2)^{1/3}}, \tag{A1}$$

with

$$\omega(0) + \psi(0) = a, \quad -\alpha\omega(0) - \beta\psi(0) = bY.$$

Therein, α and β are two distinct positive numbers, and $Y = (\omega(0)^2 + \psi(0)^2)^{1/3}$, $a = w_i^0(1)$ and $b = -\lambda_i(1, w^0(1))\partial_x w_i^0(1)$ with $\|w^0\|_{C^1([0,1])^n} < \delta$. We now show that under appropriate choice of α and β, $\omega(0)$ and $\psi(0)$ can be chosen as continuous functions of a and b. Indeed, consider the equation $P_{a,b}(Y) = 0$, where

$$P_{a,b}(Y) := (\alpha - \beta)^2 Y^3 - 2b^2 Y^2 - 2ab(\alpha + \beta)Y - a^2(\alpha^2 + \beta^2).$$

We consider $P_{a,b}(Y) = 0$ in four cases.

Case 1: $a = b = 0$. There is a unique solution $Y = 0$.

Case 2: $a = 0$, $b \neq 0$. There is a unique solution $Y = \frac{2b^2}{(\alpha-\beta)^2}$.

Case 3: $a \neq 0$, $b = 0$. There is a unique solution $Y = (\frac{a^2(\alpha^2+\beta^2)}{(\alpha-\beta)^2})^{1/3}$.

Case 4: $a \neq 0$, $b \neq 0$. Let $Y = Z + \frac{2b^2}{3(\alpha-\beta)^2}$. We have

$$Z^3 + pZ + q = 0, \tag{A2}$$

where

$$p = -\frac{2}{(\alpha - \beta)^2}\left(ab(\alpha + \beta) + \frac{2b^4}{3(\alpha - \beta)^2}\right),$$
$$q = -\frac{1}{(\alpha - \beta)^2}\left(\frac{16b^6}{27(\alpha - \beta)^4} + \frac{4ab^3(\alpha + \beta)}{3(\alpha - \beta)^2} + a^2(\alpha^2 + \beta^2)\right).$$

Let

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27} = \frac{a^4}{108(\alpha - \beta)^6}\left[16\left(\frac{b^3}{a}\right)^2 + 8(\alpha + \beta)(5\alpha^2 + 5\beta^2 - 8\alpha\beta)\frac{b^3}{a} + 27(\alpha - \beta)^2(\alpha^2 + \beta^2)^2\right].$$

For x in $\mathbb{R}$, denote

$$\tilde{\Delta}(x) = 16x^2 + 8(\alpha + \beta)(5\alpha^2 + 5\beta^2 - 8\alpha\beta)x + 27(\alpha - \beta)^2(\alpha^2 + \beta^2)^2.$$

We have

$$\Delta' = [8(\alpha+\beta)(5\alpha^2+5\beta^2-8\alpha\beta)]^2-4\cdot16\cdot27(\alpha-\beta)^2(\alpha^2+\beta^2)^2 = -128(\alpha^2-4\alpha\beta+\beta^2)^3.$$

Therefore, $\Delta > 0$ for all a, $b \neq 0$ if $\alpha^2 - 4\alpha\beta + \beta^2 > 0$. By Cardano's formula, (A2) has only one real root

$$Z = \sqrt[3]{u_1} + \sqrt[3]{u_2},$$

with

$$u_{1,2} = -\frac{q}{2} \pm \sqrt{\Delta},$$

if $\alpha^2 - 4\alpha\beta + \beta^2 > 0$. Noticing that $q < 0$ for all a, $b \neq 0$, we have $Z > 0$.

Consequently, the equation $P_{a,b}(Y) = 0$ has a unique positeve solution

$$Y = \frac{2b^2 + \sqrt[3]{\hat{u}_1} + \sqrt[3]{\hat{u}_2}}{3(\alpha-\beta)^2} \tag{A3}$$

with

$$\begin{aligned}\hat{u}_{1,2} &= 8b^6 + 18ab^3(\alpha+\beta)(\alpha-\beta)^2 + \frac{27}{2}a^2(\alpha^2+\beta^2)(\alpha-\beta)^4 \pm \frac{3\sqrt{3}}{2}(\alpha-\beta)^3\sqrt{\hat{\Delta}},\\ \hat{\Delta} &= 16a^2b^6 + 8a^3b^3(\alpha+\beta)(5\alpha^2+5\beta^2-8\alpha\beta) + 27a^4(\alpha^2+\beta^2)^2(\alpha-\beta)^2,\end{aligned}$$

if $\alpha^2 - 4\alpha\beta + \beta^2 > 0$.

Noticing that (A3) is compatible with cases 1, 2, and 3, we have that Y is continuous with respect to a and b.

Then we need to take suitable α and β such that $\zeta(t) = 0$ for $t \geq \tau_0/2$ and $\|\zeta\|_{C^1([0,\infty))} \leq C\|w^0\|_{C^1([0,1])^n}$. Without loss of generality, we assume $\beta > \alpha > 0$. Let $k_0 > 2 + \sqrt{3}$ be fixed. Take $\beta = k_0\alpha$ so that $\alpha^2 - 4\alpha\beta + \beta^2 > 0$. Denote $\Phi(t) = \omega(t)^2 + \psi(t)^2$ for $t \geq 0$. It follows from (A1) that

$$\Phi'(t) = -2\alpha\frac{\omega(t)^2 + k_0\psi(t)^2}{\Phi(t)^{1/3}} \leq -2\alpha\Phi(t)^{2/3}.$$

A direct calculation gives that $\Phi(t) = 0$ for $t \geq 3\Phi(0)^{1/3}/(2\alpha) = 3Y/(2\alpha)$. Therefore, $\zeta(t) = \omega(t) + \psi(t) = 0$ for $t \geq \tau_0/2$ if we take α large enough such that

$$\frac{3Y}{\alpha} \leq \tau_0. \tag{A4}$$

In order to estimate the dependency of α on $\|w^0\|_{C^1([0,1])^n}$ and τ_0, we need to estimate the dependency of Y on α, a and b. Using the inequality

$$\left(\sum_{i=1}^{k} y_i\right)^{\frac{1}{m}} \leq \sum_{i=1}^{k} y_i^{\frac{1}{m}}, \quad \forall m, k \in \mathbb{N}^+,\ y_1, \cdots, y_k \in \mathbb{R}^+,$$

we obtain from (A3) that there exists a constant $C_{k_0} > 0$, depending only on k_0, such that

$$Y \le C_{k_0}\left(|a|^{\frac{2}{3}} + \frac{|ab|^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}} + \frac{|a|^{\frac{1}{3}}|b|}{\alpha} + \frac{b^2}{\alpha^2}\right). \tag{A5}$$

Recalling $a = w_i^0(1)$ and $b = -\lambda_i(1, w^0(1))\partial_x w_i^0(1)$ with $\|w^0\|_{C^1([0,1])^n} < \delta$ and (19), we have that there exists a constant $C_{\delta,k_0} > 0$, depending only on δ and k_0, such that

$$\frac{Y}{\alpha} \le C_{\delta,k_0}\tilde{P}\left(\frac{\|w^0\|^{\frac{2}{3}}_{C^1([0,1])^n}}{\alpha}\right),$$

with $\tilde{P}(y) = y + y^{3/2} + y^2 + y^3$. Combined with (A4), we have that $\zeta(t) = \omega(t) + \psi(t) = 0$ for $t \ge \tau_0/2$ if

$$\alpha \ge \|w^0\|^{\frac{2}{3}}_{C^1([0,1])^n} \max\left\{\frac{12C_{\delta,k_0}}{\tau_0}, 1\right\}. \tag{A6}$$

Take α as the right-hand side of (A6). Now we estimate the C^1-norm of ζ. Since $\Phi(t)$ is monotonically decreasing as t increases, it follows from (A5) and (A6) that there exists a constant $C'_{\delta,k_0} > 0$, depending only on δ and k_0, such that for all $t \ge 0$,

$$\begin{aligned}
&|\zeta(t)|^2 \le 2\Phi(t) \le 2Y^3 \le C'_{\delta,k_0}\|w^0\|^2_{C^1([0,1])^n},\\
&|\zeta'(t)|^2 \le 2k_0^2\alpha^2\Phi(t)^{1/3} \le 2k_0^2\alpha^2 Y \le C'_{\delta,k_0}(1+\tau_0^{-2})\|w^0\|^2_{C^1([0,1])^n}.
\end{aligned}$$

Similarly, one can build the dynamics for η. We now have $a = 1$ and $b = 0$. We write $\eta = \tilde{\omega} + \tilde{\psi}$ where $\tilde{\omega}$ and $\tilde{\psi}$ satisfy the dynamics

$$\tilde{\omega}'(t) = -\frac{\tilde{\alpha}\tilde{\omega}(t)}{(\tilde{\omega}(t)^2 + \tilde{\psi}(t)^2)^{1/3}}, \quad \tilde{\psi}'(t) = -\frac{\tilde{\beta}\tilde{\psi}(t)}{(\tilde{\omega}(t)^2 + \tilde{\psi}(t)^2)^{1/3}},$$

with

$$\tilde{\omega}(0) + \tilde{\psi}(0) = 1, \quad -\tilde{\alpha}\tilde{\omega}(0) - \tilde{\beta}\tilde{\psi}(0) = 0.$$

Take $\tilde{\beta} = k_0\tilde{\alpha}$ with $k_0 > 2 + \sqrt{3}$. We have

$$\tilde{Y} = (\tilde{\omega}(0)^2 + \tilde{\psi}(0)^2)^{\frac{1}{3}} = \left(\frac{k_0^2 + 1}{(k_0 - 1)^2}\right)^{\frac{1}{3}}.$$

Similarly, $\eta(t) = \tilde{\omega}(t) + \tilde{\psi}(t) = 0$ for $t \ge \tau_0/2$ if

$$\tilde{\alpha} \ge \frac{3\tilde{Y}}{\tau_0} = 3\left(\frac{k_0^2 + 1}{(k_0 - 1)^2}\right)^{\frac{1}{3}}\tau_0^{-1}. \tag{A7}$$

Take $\tilde{\alpha}$ as the right-hand side of (A7). We have that for all $t \ge 0$,

$$|\eta(t)|^2 \le 2\tilde{Y}^3 = \frac{2k_0^2 + 2}{(k_0 - 1)^2}, \quad |\eta'(t)|^2 \le 2k_0^2\tilde{\alpha}^2\tilde{Y} = \frac{18k_0^2(k_0^2 + 1)}{(k_0 - 1)^2}\tau_0^{-2}.$$

Now we obtain the dynamics for ζ and η such that ζ and η satisfy (33) and (34).

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