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Finite-time output regulation for time-varying 2×2 hyperbolic systems

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This work considers the output regulation problem for an infinite-dimensional system. Specifically, a regulator for boundary-controlled time-varying 2×2 hyperbolic systems is designed. The output regulation is solved in finite time. Moreover, the disturbances can act within the space domain, and affect both boundaries and the to-be-controlled output. The to-be-controlled output is pointwise and may be located either at the boundary or at any point inside of the space domain. The feedback gain is a time-dependent function rather than a constant vector. Time-varying setting also brings an advantage to the problem, since the solvability of the regulator equations does not depend on the eigenmodes of the signal model. In this work, the regulator equations have a solution for any signal model. The finite-time reference signal and disturbance observer are also designed. The output feedback regulator consists of the state feedback regulator and the finite-time observer. A numerical example illustrates the regulator design and the finite-time regulation.

Keywords: output regulation; hyperbolic systems; time-varying systems; observer design.

1. Introduction

A classical problem in control theory is the output regulation problem. It consists of designing a feedback control such that the output of the system tracks a given reference and rejects the disturbances. If these exogenous signals can be modelled by a known signal model, there is a systematic solution to the output regulation problem for a linear finite-dimensional system. The output regulation for linear finite-dimensional systems is well-understood and is well introduced in, for example, Saberi *et al.* (2000). One successful method traces back to the work Davison (1976), where the signal model is included in the controller dynamics and the controller is driven by the output tracking error. Therefore, in this method the output to be controlled is assumed to be measured. The main advantage of this method is its embedded robustness. Specifically, robustness in the output regulation problem is with respect to small perturbations of system parameters. More recently, some works focusing on output regulation for nonlinear finite-dimensional systems include Astolfi *et al.* (2022); Bin *et al.* (2023).

The output regulation problem for infinite-dimensional systems, especially for partial differential equations (PDEs) system has received much attention in recent years. There has been a very fruitful literature on the output regulation problem for the first-order hyperbolic system. In Anfinsen & Aamo (2015); Anfinsen *et al.* (2017), a boundary disturbance rejection for linear 2×2 hyperbolic systems was considered by using the backstepping approach. Concerning robust output regulation, Deutscher (2017b) used backstepping method to design a robust state feedback regulator for boundary-controlled linear 2×2 hyperbolic systems with spatially varying coefficients. The disturbance can act within the domain, affecting both boundaries and the output to be controlled. Furthermore, the output to be controlled is assumed to be available for measurement. Therefore, the regulator design is based on the internal model principle. Later on, Deutscher & Gabriel (2018) generalized this work into general $n \times n$ linear heterodirectional hyperbolic systems, where the so-called p-copy internal model principle has to be fulfilled in order to achieve the robust output regulation.

Another important work on output regulation for hyperbolic systems is Deutscher (2017a), where finite-time output regulation was achieved for boundary controlled linear 2×2 hyperbolic systems by using the backstepping method. The output to be controlled does not need to be measurable. Therefore, the internal model principle cannot be applied. However, the design of an observer-based feedforward controller is still possible. The solution of the state feedback regulator problem is based on the solvability of the corresponding regulator equations which are described by ordinary differential equations (ODEs). The solvability condition is determined for the regulator equations in terms of the plant transfer behavior. The finite-time output regulation problem is achieved by the finite-time state feedback regulator, the finite-time reference observer and the finite-time disturbance observer. Moreover, Deutscher (2017c); Deutscher & Gabriel (2020) achieved finite-time output regulation for general $n \times n$ hyperbolic systems with different convergent times. Other works focusing on output regulation of other types of PDEs include Lhachemi *et al.* (2021); Guo & Zhao (2022) for heat equation, Balogoun *et al.* (2023) for Korteweg-de Vries equation, Guo & Meng (2021) for beam equation and Liu *et al.* (2022) for thermoelastic system.

Concerning the output regulation for non-autonomous system, Ichikawa & Katayama (2006) provided general conditions under which the output regulation problem for finite-dimensional system can be solved. In Zhang & Serrani (2009), robust output regulation for linear periodic system was solved. For the output regulation for infinite-dimensional system, Paunonen (2017) consider the output regulation problem for continuous-time periodic linear systems with periodic reference and disturbance signals.

In this paper, we consider the finite-time output regulation problem for linear time-varying 2×2 hyperbolic system. Therein, the disturbances can act within the domain, affecting both boundaries and the output to be controlled. The output to be controlled may be located at a boundary or is defined pointwise in-domain. Different from the system in Deutscher (2017a), the transport velocity, boundary coupling coefficient and disturbance input locations are time dependent in our system. This brings new difficulties and challenges in the output regulation problem. The regulator design is based on the solvability of the corresponding regulator equations which are PDEs rather than ODEs as in Deutscher (2017a). Therefore, the feedback gain is a time dependent function rather than a constant vector. In the meanwhile, time-varying settings also bring some advantages. Since the solution of regulator equations is time dependent, the solvability of regulator equations depends no longer on the relationship between the plant transfer behavior and the eigenmodes of the signal model. In this work, the regulator equations are solvable for any signal model. In contrast, the regulator equations in Deutscher (2017a) are solvable if the eigenmodes of the signal model are not blocked by the transfer behavior of the time-independent hyperbolic system. For observer design, due to the time-varying nature of the system, we employ the concept of uniform observability (introduced in Menold *et al.* (2003)) and provide a constructive observer design under this assumption. Moreover, we present a computationally feasible method for implementing the observer. Finally, we validate our theoretical results with a numerical example, demonstrating the effectiveness of our approach. The main contribution of this work can be summarized as follows. First, we solve the finite-time output regulation problem for time-varying hyperbolic systems, which introduces significant challenges due to the time-dependent PDE structure of the regulator equation. Second, we derive explicit expressions for both the feedback control law and the observer design, and based on these results, we provide a numerical example to support our theoretical findings. In our related work Bai *et al.* (2025), we study a more general setting of time-varying hyperbolic systems. However, due to the complexity of the output operator and the presence of zeroth-order terms, Bai *et al.* (2025) only establishes the existence of a solution to the regulator equation without providing an explicit expression or numerical simulations. In contrast, the current paper offers constructive solutions and numerical validation, making it more applicable in practice.

This paper is organized as follows. In Section 2, we describe the linear time-varying hyperbolic system that we consider and we give the definition of the regulation problem. Section 3 provides some physical examples and some preliminaries about the characteristics associated with linear time-varying hyperbolic systems. We state and prove the first main result, the state feedback regulator, in Section 4. The finite-time reference signal and disturbance observer are introduced in Section 5. The second main result, the output feedback regulator, is also proven in Section 5. A numerical simulation is given in Section 6. Concluding remarks are collected in Section 7.

2. Problem statement

Consider the following linear time-varying 2×2 hyperbolic system,

$$\partial_t w(t,x) = \Lambda(t,x)\partial_x w(t,x) + g_1(t,x)d_1(t), \tag{2.1a}$$

$$w_1(t,0) = q_0(t)w_2(t,0) + g_2(t)d_2(t), \tag{2.1b}$$

$$w_2(t,1) = q_1(t)w_1(t,1) + u(t) + g_3(t)d_3(t), \tag{2.1c}$$

$$w(t_0,x) = w^0(x), \tag{2.1d}$$

$$y(t) = w_1(t,x_r) + g_4(t)d_4(t), \tag{2.1e}$$

$$y_m(t) = w_2(t,0). \tag{2.1f}$$

In Equation (2.1), (t,x) is in $\mathscr{D}(t_0) := \{(t,x) | t > t_0, 0 < x < 1\}$ with $t_0 \geq 0$, $w(t,x) = (w_1,w_2)^\top(t,x)$ in $\mathbb{R}^2$ is the state, initial data w^0 at time t_0 is assumed to be piecewise continuous, $u(t)$ in $\mathbb{R}$ is the control input, $d_i(t)$ in $\mathbb{R}$, $i = 1,2,3,4$, are disturbances, and $g_1(t,x)$ in $\mathbb{R}^2$, $g_i(t)$ in $\mathbb{R}$, $i = 2,3,4$, are corresponding disturbance input locations. The output to be controlled $y(t)$ in $\mathbb{R}$ is a single boundary or pointwise output with x_r in $[0,1]$. The available measurement $y_m(t)$ is in $\mathbb{R}$. The functions q_0, q_1 : $[0,\infty) \to \mathbb{R}$ couple the equations of the system on the boundaries $x = 0$ and $x = 1$. Let us make the following assumptions on all coefficients involved in Equation (2.1).

Assumption 1. The matrix Λ is diagonal, namely $\Lambda(t,x) = \mathrm{diag}(-\lambda_1(t,x), \lambda_2(t,x))$ for every $t \geq 0$ and $0 \leq x \leq 1$.

Assumption 2. The functions λ_1, λ_2 and q_0 are uniformly away from zero, i.e. there exists $\varepsilon_0 > 0$ such that for every $t \geq 0$ and $0 \leq x \leq 1$,

$$\lambda_1(t,x) > \varepsilon_0 > 0, \quad \lambda_2(t,x) > \varepsilon_0 > 0, \tag{2.2}$$

and there exists $\varepsilon_q > 0$ such that for every $t \geq 0$,

$$|q_0(t)| > \varepsilon_q > 0. \tag{2.3}$$

Assumption 2 will be commented in Remark 2 below.

ASSUMPTION 3. The following regularities hold for Λ, q_0 and q_1:

$$\Lambda \in C^1(\mathbb{R}^2)^{2\times 2}, \quad q_0, q_1 \in C^0([0,\infty)), \quad \Lambda, \partial_x \Lambda \in L^\infty(\mathbb{R}^2)^{2\times 2}, \quad q_0, q_1 \in L^\infty(0,\infty).$$

Here Λ is defined in $\mathbb{R}^2$. The value of Λ outside $\overline{\mathscr{D}(0)}$ is only used when defining characteristics (see Section 3.2).

The functions $d_i : [0,\infty) \to \mathbb{R}$, $i = 1,2,3,4$, are disturbances. The corresponding disturbance input locations satisfy the following assumption.

ASSUMPTION 4. Functions g_i, $i = 1,2,3,4$ are known and have the following regularities

$$g_1 \in C^0([0,\infty) \times [0,1])^2 \cap L^\infty((0,\infty) \times (0,1))^2, \quad g_i \in C^0([0,\infty)) \cap L^\infty(0,\infty), \quad i = 2,3,4.$$

The disturbances and the reference input $r(t)$ in $\mathbb{R}$ to be asymptotically tracked by the output $y(t)$ can be represented by the solutions of the finite-dimensional signal model, for $t \geq t_0 \geq 0$,

$$\dot{v}(t) = Sv(t), \quad v(t_0) = v^0 \in \mathbb{R}^{n_v}, \tag{2.4a}$$

$$r(t) = p_r^\top v_r(t), \quad d_i(t) = p_i^\top v_d(t), \quad i = 1,2,3,4, \tag{2.4b}$$

where S in $\mathbb{R}^{n_v \times n_v}$ is a block diagonal matrix, i.e. $S = \mathrm{bdiag}(S_d, S_r)$, and $v = (v_d^\top, v_r^\top)^\top$ with v_d in $\mathbb{R}^{n_d}$ and v_r in $\mathbb{R}^{n_r}$, $n_d + n_r = n_v$. Therefore, we have the n_dth order disturbance model $\dot{v}_d(t) = S_d v_d(t)$, $v_d(t_0) = v_d^0$, and the n_rth order reference model $\dot{v}_r(t) = S_r v_r(t)$, $v_r(t_0) = v_r^0$. For the design of the regulator, it is assumed that S, p_r and p_i, $i = 1,2,3,4$ are known. For the observer design, it is assumed that only the reference input r and the measurement y_m are known.

This paper concerns the uniform finite-time output regulation problem. Denote by

$$e_y(t) = y(t) - r(t) \tag{2.5}$$

the output tracking error. Let us give the definition of the regulation that we are interested.

DEFINITION 1. Let $T_0 > 0$ and let the disturbances d_i, $i = 1,2,3,4$, and the reference input r is generated by the finite-dimensional signal model (2.4). We say the output y of the system (2.1) achieves the uniform finite-time output regulation with settling time T_0 by state feedback regulator (resp. by output feedback regulator) if there exists a feedback control $u = \mathscr{K}_s[w, v]$ (resp. $u = \mathscr{K}_o[y_m, r]$) such that for all $t_0 \geq 0$, v^0 in $\mathbb{R}^{n_v}$ and piecewise continuous w^0, the output tracking error e_y satisfies $e_y(t) = 0$ for $t \geq t_0 + T_0$.

REMARK 1. The uniformity means that the output regulation is achieved uniformly to the initial time t_0.

3. Preliminaries

3.1. *Motivation of the problem: some physical models*

As mentioned in Bastin & Coron (2016); Coron *et al.* (2021), time-varying hyperbolic systems appear in the linearized Saint-Venant equations and many other physical models of balance laws. In this section, we give some examples that can be modelled by linear time-varying hyperbolic systems.

EXAMPLE 1 (Mining cable elevator with flexible guides Wang & Krstic (2022)). The lateral vibration control of a mining cable elevator with viscoelastic guides can be modelled as a 2×2 hyperbolic system coupled with an ODE, given by

$$\begin{aligned}
\dot{X}(t) &= AX(t) + Bz_2(t,0) + B_1 d(t),\\
z_1(t,0) &= CX(t) - p_1 z_2(t,0),\\
\partial_t z_1(t,y) &= -q_1(y)\partial_y z_1(t,y) + c_1(y)z_1(t,y) + c_2(y)z_2(t,y),\\
\partial_t z_2(t,y) &= q_2(y)\partial_y z_2(t,y) + c_3(y)z_1(t,y) + c_4(y)z_2(t,y),\\
z_2(t,l(t)) &= U(t) + p_2 z_1(t,l(t)),
\end{aligned}$$

with y in $[0, l(t)]$ and t in $[0, \infty)$. The precise physical meanings of the variables in the equation can be found in (Wang & Krstic, 2022, Chap. 5). This equation is a moving-boundary hyperbolic system. Through a coordinate transformation $x = y/l(t)$, the PDE-subsystem can be converted into a hyperbolic system with time-dependent speeds. More precisely, let $x = y/l(t)$, $w_1(t,x) = z_1(t,y)$ and $w_2(t,x) = z_2(t,y)$. Direct calculation shows that the PDE-subsystem is equivalent to

$$\begin{aligned}
\partial_t w_1(t,x) &= -\frac{q_1(l(t)x) - l'(t)x}{l(t)}\partial_x w_1(t,x) + c_1(l(t)x)w_1(t,x) + c_2(l(t)x)w_2(t,x),\\
\partial_t w_2(t,x) &= \frac{q_2(l(t)x) + l'(t)x}{l(t)}\partial_x w_2(t,x) + c_3(l(t)x)w_1(t,x) + c_4(l(t)x)w_2(t,x),\\
w_1(t,0) &= CX(t) - p_1 w_2(t,0),\\
w_2(t,1) &= U(t) + p_2 w_1(t,1),
\end{aligned}$$

with x in $[0,1]$ and t in $[0,\infty)$. Note that Assumptions 5.3 and 5.4 in (Wang & Krstic, 2022, Chap. 5) ensures that w-system is strictly hyperbolic and maintains different propagation directions.

EXAMPLE 2 (Plug flow chemical reactors Bastin & Coron (2016)). The dynamics of the plug flow chemical reactors are then described by the following semi-linear system of balance laws:

$$\begin{aligned}
&\partial_t T_c(t,x) - V_c(t)\partial_x T_c(t,x) - k_o(T_c(t,x) - T_r(t,x)) = 0,\\
&\partial_t T_r(t,x) + V_r(t)\partial_x T_r(t,x) + k_o(T_c(t,x) - T_r(t,x)) - k_1 r(T_r(t,x), C_A(t,x), C_B(t,x)) = 0,\\
&\partial_t C_A(t,x) + V_r(t)\partial_x C_A(t,x) + r(T_r(t,x), C_A(t,x), C_B(t,x)) = 0,\\
&\partial_t C_B(t,x) + V_r(t)\partial_x C_B(t,x) - r(T_r(t,x), C_A(t,x), C_B(t,x)) = 0.
\end{aligned}$$

Then the system is a semi-linear system of balance laws with time-dependent speeds. The precise physical meanings of the variables in the equation can be found in (Bastin & Coron, 2016, Chap. 1.7).

3.2. *Preliminaries on characteristics*

In this section, we introduce some known facts on the characteristics associated with linear time-varying hyperbolic system (see Coron *et al.* (2021)). Let χ_1 and χ_2 be the flow associated with λ_1 and $-\lambda_2$ respectively, namely for every (t,x) in $\mathbb{R}^2$, the functions $s \mapsto \chi_1(s;t,x)$ and $s \mapsto \chi_2(s;t,x)$ are the solution to the ODE, for s in $\mathbb{R}$,

$$\frac{\partial}{\partial s}\chi_1(s;t,x) = \lambda_1(s,\chi_1(s;t,x)), \quad \chi_1(t;t,x) = x, \tag{3.1}$$

and, for s in $\mathbb{R}$,

$$\frac{\partial}{\partial s}\chi_2(s;t,x) = -\lambda_2(s,\chi_2(s;t,x)), \quad \chi_2(t;t,x) = x, \tag{3.2}$$

respectively. The existence and uniqueness of the solutions to the ODEs (3.1) and (3.2) follow the classical theory. Moreover, due to Assumption 3, the solutions are global and have the regularity

$$\chi_1, \chi_2 \in C^1(\mathbb{R}^3). \tag{3.3}$$

Next we introduce the entry and exit times for the interval $[0,1]$. For $j = 1,2$, t in $\mathbb{R}$ and x in $[0,1]$, let $s_j^{\text{in}}(t,x)$ and $s_j^{\text{out}}(t,x)$ be the entry and exit times of the flow $\chi_j(\cdot;t,x)$ inside the interval $[0,1]$, namely the respective unique solutions to

$$\chi_1(s_1^{\text{in}}(t,x);t,x) = 0, \quad \chi_1(s_1^{\text{out}}(t,x);t,x) = 1, \quad \chi_2(s_2^{\text{in}}(t,x);t,x) = 1, \quad \chi_2(s_2^{\text{out}}(t,x);t,x) = 0.$$

The existence and uniqueness of $s_j^{\text{in}}(t,x)$ and $s_j^{\text{out}}(t,x)$ are guaranteed by (2.2) in Assumption 2.2. From (3.3) and by the implicit function theorem, we have

$$s_j^{\text{in}}, s_j^{\text{out}} \in C^1(\mathbb{R}\times[0,1]), \quad j = 1,2. \tag{3.4}$$

Integrating the ODEs (3.1) and (3.2) and using the assumption (2.2), we have the following bounds for every t in $\mathbb{R}$ and x in $[0,1]$,

$$t - s_j^{\text{in}}(t,x) < \frac{1}{\varepsilon_0}, \quad s_j^{\text{out}}(t,x) - t < \frac{1}{\varepsilon_0}, \quad j = 1,2. \tag{3.5}$$

Similar to the definition of $s_j^{\text{out}}(t,x)$, let $s_1^r(t)$ be the exit time of the flow $\chi_1(\cdot;t,0)$ inside the interval $[0,x_r]$, namely the unique solution to $\chi_1(s_1^r(t);t,0) = x_r$. Especially, $s_1^r(t) = s_1^{\text{out}}(t,0)$ if $x_r = 1$.

4. Finite-time output regulation by state feedback regulator

In this section, we aim to find a finite-time state feedback regulator. We consider the following time dependent regulator,

$$u(t) = -q_1(t)w_1(t,1) + k_v(t)^\top v(t), \tag{4.1}$$

with feedback gain function $k_v : [0,\infty) \to \mathbb{R}^{n_v}$ to be determined later. By applying (4.1) to the system (2.1) and taking the signal model (2.4) into account, we have the closed-loop system, for $t \geq t_0$, $0 \leq x \leq 1$,

$$\dot{v}(t) = Sv(t), \quad v(t_0) = v^0, \tag{4.2a}$$

$$\partial_t w(t,x) = \Lambda(t,x)\partial_x w(t,x) + \tilde{g}_1(t,x)v(t), \tag{4.2b}$$

$$w_1(t,0) = q_0(t)w_2(t,0) + \tilde{g}_2(t)^\top v(t), \tag{4.2c}$$

$$w_2(t,1) = k_v(t)^\top v(t) + \tilde{g}_3(t)^\top v(t), \tag{4.2d}$$

$$w(t_0,x) = w^0(x), \tag{4.2e}$$

$$e_y(t) = w_1(t,x_r) - (\tilde{p}_r - \tilde{g}_4(t))^\top v(t), \tag{4.2f}$$

where $\tilde{p}_r^\top = (0_{n_d}^\top, p_r^\top)$, $\tilde{g}_1(t,x) = g_1(t,x)(p_1^\top, 0_{n_r}^\top)$, $\tilde{g}_i(t)^\top = g_i(t)(p_i^\top, 0_{n_r}^\top)$, $i = 2,3,4$, and e_y is defined in (2.5). By the classical method of characteristics (see (Coron *et al.*, 2021, Theorem A.2, Remark A.4)), one can prove the following well-posedness result for (4.2b) to (4.2e).

THEOREM 1. Under Assumptions 1 to 4 and k_v in $C^0([0,\infty))^{n_v} \cap L^\infty(0,\infty)^{n_v}$, for every $t_0 \geq 0$, v^0 in $\mathbb{R}^{n_v}$ and piecewise continuous initial data w^0, there exists a unique solution w to system (4.2b) to (4.2e) which is piecewise continuous in $\mathscr{D}(t_0)$.

Let us now state the first main result of this paper.

THEOREM 2. Let Assumptions 1 to 4 hold. Let settling time $T_{\text{unif}}(\Lambda)$ be defined by

$$T_{\text{unif}}(\Lambda) = \sup_{t\geq 0}[s_1^{\text{out}}(s_2^{\text{out}}(t,1),0) - t]. \tag{4.3}$$

Choose feedback control $u(t) = -q_1(t)w_1(t,1) + k_v(t)^\top v(t)$, where feedback gain function k_v in $C^0([0,\infty))^{n_v} \cap L^\infty(0,\infty)^{n_v}$ is defined as follows, for all $t \geq 0$,

$$\begin{aligned} k_v(t)^\top = {} & -\tilde{g}_3(t)^\top - \int_t^{s_2^{\text{out}}(t,1)} \tilde{g}_{12}(s,\chi_2(s;t,1))\mathrm{e}^{(s-t)S}\mathrm{d}s + q_0(s_2^{\text{out}}(t,1))^{-1}\Big[-\tilde{g}_2(s_2^{\text{out}}(t,1))^\top \mathrm{e}^{(s_2^{\text{out}}(t,1)-t)S} \\ & + [\tilde{p}_r - \tilde{g}_4(s_1^r(s_2^{\text{out}}(t,1)))]^\top \mathrm{e}^{(s_1^r(s_2^{\text{out}}(t,1))-t)S} - \int_{s_2^{\text{out}}(t,1)}^{s_1^r(s_2^{\text{out}}(t,1))} \tilde{g}_{11}(s,\chi_1(s;s_2^{\text{out}}(t,1),0))\mathrm{e}^{(s-t)S}\mathrm{d}s\Big]. \end{aligned} \tag{4.4}$$

Then, for every $t_0 \geq 0$, v^0 in $\mathbb{R}^{n_v}$ and piecewise continuous initial data w^0, there exists a unique solution w to system (4.2b) to (4.2e) which is piecewise continuous in $\mathscr{D}(t_0)$. Moreover, the output y of the system

(2.1) achieves the uniform finite-time output regulation with settling time $T_{\text{unif}}(\Lambda)$ by state feedback regulator.

For any initial time t_0, the time cost $s_1^{\text{out}}(s_2^{\text{out}}(t_0,1),0)-t_0$ is the time to transport along $\chi_2(\cdot;t_0,1)$ from $x=1$ to $x=0$ and then along $\chi_1(\cdot;s_2^{\text{out}}(t_0,1),0)$ from $x=0$ to $x=1$. Therefore, the settling time $T_{\text{unif}}(\Lambda)$ is the uniform time cost with respect to the initial time t_0. Let us remark that thanks to (3.5), it holds $0<T_{\text{unif}}(\Lambda)<2/\varepsilon_0$.

REMARK 2. Equation (2.2) in Assumption 2 is identical to the assumption in Coron *et al.* (2021), where finite-time stabilization problem is considered. As stated in Coron *et al.* (2021), (2.2) is expected for finite-time stabilization. For the output regulation problem, (2.3) is needed in finding the feedback gain function k_v (see (4)). Therefore, the conditions for the uniform finite-time output regulation are stronger than the conditions for the finite-time stabilization.

We prove Theorem 2 in two steps. In the first step, we use a change of coordinates to transform the system into the error system. Then we prove the error system is finite-time convergent in settling time $T_{\text{unif}}(\Lambda)$, and thus, the output tracking error converges in settling time $T_{\text{unif}}(\Lambda)$. The second step is to prove the existence of the change of coordinates. It is formulated by the solvability of the regulator equations. The feedback gain function k_v is determined by the solution to regulator equation (4.6).

4.1. *Removal of the dependency of v*

In the first step of designing the regulator, we introduce a bounded invertible change of coordinates to eliminate the dependency of v in Equations (4.2b) to (4.2f),

$$z(t,x)=w(t,x)-\Pi(t,x)v(t),$$

with $\Pi=[\Pi_{ij}]:[0,\infty)\times[0,1]\to\mathbb{R}^{2\times n_v}$. Then (4.2) takes the form, for $t\geq t_0$, $0\leq x\leq 1$,

$$\dot v(t)=Sv(t),\quad v(t_0)=v^0, \tag{4.5a}$$

$$\partial_t z(t,x)=\Lambda(t,x)\partial_x z(t,x), \tag{4.5b}$$

$$z_1(t,0)=q_0(t)z_2(t,0), \tag{4.5c}$$

$$z_2(t,1)=0 \tag{4.5d}$$

$$z(t_0,x)=w^0(x)-\Pi(t_0,x)v^0, \tag{4.5e}$$

$$e_y(t)=z(t,x_r), \tag{4.5f}$$

if Π is the solution to the regulator equations, for $t\geq 0$, $0\leq x\leq 1$,

$$\partial_t\Pi(t,x)=\Lambda(t,x)\partial_x\Pi(t,x)-\Pi(t,x)S+\tilde g_1(t,x), \tag{4.6a}$$

$$\Pi_1(t,0)=q_0(t)\Pi_2(t,0)+\tilde g_2(t)^\top, \tag{4.6b}$$

$$\Pi_1(t,x_r)=(\tilde p_r-\tilde g_4(t))^\top, \tag{4.6c}$$

and

$$k_v(t)^\top = \Pi_2(t,1) - \tilde{g}_3(t)^\top, \tag{4.7}$$

where $\Pi_i(t,x) = (\Pi_{i1}, \cdots, \Pi_{in_v})(t,x)$ belongs to $\mathbb{R}^{1\times n_v}$, $i = 1, 2$. The finite-time stability of z-subsystem (4.5b) to (4.5d) with settling time $T_{\mathrm{unif}}(\Lambda)$ defined by (4.3) is a corollary of (Coron *et al.*, 2021, Prop. 2.12).

Therefore, the uniform finite-time output regulation problem is solved if there exists a solution Π of the regulator equation (4.6).

4.2. *Solvability of the regulator equations*

In this section, we prove the solvability of the regulator equation (4.6). Postmultiply (4.6) by e^{tS}, the matrix exponential of tS, and denote $\hat{\Pi}(t,x) = \Pi(t,x)\mathrm{e}^{tS}$. This yields the following equations, for $t \geq 0$, $0 \leq x \leq 1$,

$$\partial_t \hat{\Pi}(t,x) = \Lambda(t,x)\partial_x \hat{\Pi}(t,x) + \hat{g}_1(t,x), \tag{4.8a}$$

$$\hat{\Pi}_1(t,0) = q_0(t)\hat{\Pi}_2(t,0) + \hat{g}_2(t)^\top, \tag{4.8b}$$

$$\hat{\Pi}_1(t,x_r) = \hat{g}_4(t)^\top, \tag{4.8c}$$

where $\hat{g}_1(t,x) = \tilde{g}_1(t,x)\mathrm{e}^{tS}$, $\hat{g}_2(t)^\top = \tilde{g}_2(t)^\top \mathrm{e}^{tS}$ and $\hat{g}_4(t)^\top = (\tilde{p}_r - \tilde{g}_4(t))^\top \mathrm{e}^{tS}$.

Remark 3. The solvability of (4.8) does not depend neither on the signal matrix S nor on the initial condition v^0. Namely, for any signal matrix S and initial condition v^0, there exists a solution to (4.8). This property remains valid when considering hyperbolic systems with zeroth-order terms and more general output operators, as detailed in Bai *et al.* (2025). This property is essentially different from the time independent case, where the regulator equation becomes an ODE dependent on S. This ODE is solvable if and only if S satisfies a condition related to the zeros of the transfer function (see Deutscher (2017a)). Anexample from item 2 of Remark 4.3 in Bai *et al.* (2025) illustrates that a time-varying feedback regulator may still exist even when the condition of Deutscher (2017a) is not satisfied.

We use characteristics to solve the regulator equations. Notice that the regulator equation (4.8) is only coupled by the boundary condition (4.8b). Therefore, we first solve $\hat{\Pi}_1(t,0)$ from (4.8a) to (4.8c), then solve $\hat{\Pi}_2(t,1)$ from (4.8a) to (4.8b). Recalling the definitions of $s_1^r(t)$ and $s_2^{\mathrm{out}}(t,x)$ in Section 3.2, by the characteristics, from (4.8a), we obtain, for $t \geq 0$,

$$\hat{\Pi}_1(t,0) = \hat{\Pi}_1(s_1^r(t), x_r) - \int_t^{s_1^r(t)} \hat{g}_{11}(s, \chi_1(s;t,0))\mathrm{d}s, \tag{4.9}$$

where $\hat{g}_{11}$ is the first row of $\hat{g}_1$. Similarly, from (4.8a), we get that, for $t \geq 0$,

$$\hat{\Pi}_2(t,1) = \hat{\Pi}_2(s_2^{\mathrm{out}}(t,1), 0) - \int_t^{s_2^{\mathrm{out}}(t,1)} \hat{g}_{12}(s, \chi_2(s;t,1))\mathrm{d}s, \tag{4.10}$$

where $\hat{g}_{12}$ is the second row of $\hat{g}_1$. Consequently, it follows from (4.8b), (4.8c), (4.9) and (4.10) that, for $t \geq 0$,

$$\hat{\Pi}_2(t,1) = -\int_t^{s_2^{\mathrm{out}}(t,1)} \hat{g}_{12}(s, \chi_2(s;t,1))\mathrm{d}s + q_0(s_2^{\mathrm{out}}(t,1))^{-1}\Bigg[\hat{g}_4(s_1^r(s_2^{\mathrm{out}}(t,1)))^\top - \hat{g}_2(s_2^{\mathrm{out}}(t,1))^\top - \int_{s_2^{\mathrm{out}}(t,1)}^{s_1^r(s_2^{\mathrm{out}}(t,1))} \hat{g}_{11}(s, \chi_1(s; s_2^{\mathrm{out}}(t,1),0))\mathrm{d}s\Bigg]. \tag{4.11}$$

Postmultiplying (11) by e^{-tS}, from (2.3), we obtain that, for $t \geq 0$,

$$\Pi_2(t,1) = -\int_t^{s_2^{\mathrm{out}}(t,1)} \tilde{g}_{12}(s, \chi_2(s;t,1))\mathrm{e}^{(s-t)S}\mathrm{d}s + q_0(s_2^{\mathrm{out}}(t,1))^{-1}\Bigg[-\tilde{g}_2(s_2^{\mathrm{out}}(t,1))^\top \mathrm{e}^{(s_2^{\mathrm{out}}(t,1)-t)S} + [\tilde{p}_r - \tilde{g}_4(s_1^r(s_2^{\mathrm{out}}(t,1)))]^\top \mathrm{e}^{(s_1^r(s_2^{\mathrm{out}}(t,1))-t)S} - \int_{s_2^{\mathrm{out}}(t,1)}^{s_1^r(s_2^{\mathrm{out}}(t,1))} \tilde{g}_{11}(s, \chi_1(s; s_2^{\mathrm{out}}(t,1),0))\mathrm{e}^{(s-t)S}\mathrm{d}s\Bigg]. \tag{4.12}$$

Therefore, the expression (4) of the feedback gain function k_v is given by (4.7) and (12). It follows from Assumptions 2 to 4, (3.4), (3.5) and (12) that k_v is continuous and bounded.

Remark 4. The solution to the regulator equation (4.8) may not be unique. In fact, when $x_r < 1$, by the characteristic method, the values of Π_1 in $\{(t,x) \in [0,\infty) \times [0,1] | x > \chi_1(t;0,x_r)\}$ can be chosen arbitrarily, while the values of Π_1 in $\{(t,x) \in [0,\infty) \times [0,1] | x \leq \chi_1(t;0,x_r)\}$ and Π_2 in $[0,\infty) \times [0,1]$ are uniquely determined by the regulator equation (4.8). Thus by (4.7), the gain function k_v is uniquely determined.

5. Finite-time observers

As mentioned in Section 2, only the reference input $r(t)$ and the measurement $y_m(t)$ are known. In this section, we design the observers for the signal state v_r, and for the state w and the disturbance state v_d, respectively.

5.1. *Finite-time reference signal observer*

The finite-time convergent observer has been introduced in the literature (see Engel & Kreisselmeier (2002); Deutscher (2017a)). Let us recall the observer design. In this section, we assume that $(p_r^\top, S_r)$ is observable. Consider two identical reference observers

$$\begin{aligned} \dot{\hat{v}}_r^j(t) &= S_r \hat{v}_r^j(t) + l_r^j(r(t) - p_r^\top \hat{v}_r^j(t)), \quad t > t_0,\ j = 1,2, \\ \hat{v}_r^j(t_0) &= \hat{v}_r^{j,0}, \end{aligned} \tag{5.1}$$

with l_r^j in $\mathbb{R}^{n_r}$ to be determined later. By assuming $F_r^j = S_r - l_r^j p_r^\top$, $j = 1,2$, and

$$F_r = \begin{bmatrix} F_r^1 & 0 \\ 0 & F_r^2 \end{bmatrix}, \quad \underline{l}_r = \begin{bmatrix} l_r^1 \\ l_r^2 \end{bmatrix}, \quad \underline{\hat{v}}_r = \begin{bmatrix} \hat{v}_r^1 \\ \hat{v}_r^2 \end{bmatrix}, \quad \underline{\hat{v}}_r^0 = \begin{bmatrix} \hat{v}_r^{1,0} \\ \hat{v}_r^{2,0} \end{bmatrix}, \tag{5.2}$$

we combine these two observers in one equation and use a delay $D_r > 0$ to generate the finite-time reference estimate $\hat{v}_r^+$ by

$$\begin{aligned}\dot{\hat{\underline{v}}}_r(t) &= F_r \hat{\underline{v}}_r(t) + \underline{l}_r r(t), \quad t > t_0, \quad \hat{\underline{v}}_r(t_0) = \hat{\underline{v}}_r^0, \\ \hat{v}_r^+(t) &= \begin{bmatrix}\mathrm{Id}_{n_r} & 0\end{bmatrix}\begin{bmatrix}\mathrm{B}_{n_r} & \mathrm{e}^{F_r D_r}\mathrm{B}_{n_r}\end{bmatrix}^{-1}(\hat{\underline{v}}_r(t) - \mathrm{e}^{F_r D_r}\hat{\underline{v}}_r(t - D_r)),\end{aligned} \tag{5.3}$$

where $\mathrm{B}_{n_r} = [I_{n_r}, I_{n_r}]^\top$. Because of the delay, this observer has initial conditions $\hat{\underline{v}}_r(t)$, t in $[t_0 - D_r, t_0]$. Without loss of generality, assume that $\hat{\underline{v}}_r(t) = \hat{\underline{v}}_r^0$ for t in $[t_0 - D_r, t_0]$. The next proposition proved in Engel & Kreisselmeier (2002); Deutscher (2017a) shows that the observer $\hat{v}_r^+$ converges to v_r in finite time D_r.

Proposition 3. Assume that $(p_r^\top, S_r)$ is observable. There exists $\underline{l}_r$ such that F_r is Hurwitz and for almost all $D_r > 0$, $\det[\mathrm{B}_{n_r} \quad \mathrm{e}^{F_r D_r}\mathrm{B}_{n_r}] \neq 0$. Then, (5.3) is an observer for v_r, whose state estimate $\hat{v}_r^+$ converges to v_r at finite time D_r. More precisely, for any $t_0 \geq 0$, v_r^0, $\hat{v}_r^{1,0}$ and $\hat{v}_r^{2,0}$ in $\mathbb{R}^{n_r}$, we have $\hat{v}_r^+(t) = v_r(t)$ for $t \geq t_0 + D_r$.

5.2. *Finite-time disturbance observer*

In this section, let us assume that Λ, q_1 and g_i, $i = 1, 2, 3$, are smooth. The reason for this assumption is that we need the finite-time convergence observer results from Menold *et al.* (2003).

The first step for designing a finite-time disturbance observer is to design an asymptotic disturbance observer. To this end, let us consider the following disturbance observer, for (t, x) in $\mathscr{D}(t_0)$,

$$\dot{\hat{v}}_d(t) = S_d \hat{v}_d(t) + l_d(t)(y_m(t) - \hat{w}_2(t, 0)), \tag{5.4a}$$

$$\partial_t \hat{w}(t, x) = \Lambda(t, x)\partial_x \hat{w}(t, x) + \hat{g}_1(t, x)\hat{v}_d(t) + l_w(t, x)(y_m(t) - \hat{w}_2(t, 0)), \tag{5.4b}$$

$$\hat{w}_1(t, 0) = q_0(t) y_m(t) + \hat{g}_2(t)^\top \hat{v}_d(t), \tag{5.4c}$$

$$\hat{w}_2(t, 1) = q_1(t)\hat{w}_1(t, 1) + u(t) + \hat{g}_3(t)\hat{v}_d(t), \tag{5.4d}$$

$$\hat{v}_d(t_0) = \hat{v}_d^0, \quad \hat{w}(t_0, x) = \hat{w}^0(x), \tag{5.4e}$$

where $\hat{v}_d$ and $\hat{w}$ are the observer states, $\hat{v}_d^0$ and $\hat{w}^0$ are the initial data, y_m is the measurement defined in (2.1f), $\hat{g}_1(t, x) = g_1(t, x)p_1^\top$, $\hat{g}_i(t)^\top = g_i(t)p_i^\top$, $i = 2, 3$, and l_d and l_w are the observer gain functions to be determined later. By introducing the observer errors $e_d = v_d - \hat{v}_d$ and $e_w = w - \hat{w}$, we have the observer error system for (t, x) in $\mathscr{D}(t_0)$,

$$\dot{e}_d(t) = S_d e_d(t) - l_d(t) e_{w,2}(t, 0), \tag{5.5a}$$

$$\partial_t e_w(t, x) = \Lambda(t, x)\partial_x e_w(t, x) + \hat{g}_1(t, x)e_d(t) - l_w(t, x)e_{w,2}(t, 0), \tag{5.5b}$$

$$e_{w,1}(t, 0) = \hat{g}_2(t)^\top e_d(t), \tag{5.5c}$$

$$e_{w,2}(t, 1) = q_1(t)e_{w,1}(t, 1) + \hat{g}_3(t)^\top e_d(t). \tag{5.5d}$$

In order to determine the gain functions l_d and l_w, we use the time-varying coordinate transformation to transform the error system (5.5) into a PDE-ODE cascade system. This method is inspired by the disturbance observer design in Deutscher (2017a). We introduce the following transformations. Let

$$\varepsilon_w(t,x) = e_w(t,x) - N(t,x)e_d(t), \tag{5.6}$$

with N defined on $(0,\infty)\times(0,1)$. By direct calculation, the system of e_d and ε_w reads, for (t,x) in $\mathscr{D}(t_0)$,

$$\dot{e}_d(t) = (S_d - l_d(t)N_2(t,0))e_d(t) - l_d(t)\varepsilon_{w,2}(t,0), \tag{5.7a}$$

$$\partial_t\varepsilon_w(t,x) = \Lambda(t,x)\partial_x\varepsilon_w(t,x), \tag{5.7b}$$

$$\varepsilon_{w,1}(t,0) = 0, \tag{5.7c}$$

$$\varepsilon_{w,2}(t,1) = q_1(t)\varepsilon_{w,1}(t,1), \tag{5.7d}$$

if N satisfies, for (t,x) in $(0,+\infty)\times(0,1)$,

$$\partial_t N(t,x) + N(t,x)S_d = \Lambda(t,x)\partial_x N(t,x) + \hat{g}_1(t,x), \tag{5.8a}$$

$$N_1(t,0) = \hat{g}_2(t)^\top, \tag{5.8b}$$

$$N_2(t,1) = q_1(t)N_1(t,1) + \hat{g}_3(t)^\top, \tag{5.8c}$$

and

$$l_w(t,x) = N(t,x)l_d(t). \tag{5.9}$$

Therein, N_1 (resp. N_2) is the first row (resp, the second row) of N. By adding an artificial initial conditions $N(0,x) = N^0(x)$ and using the change of coordinate $\hat{N}(t,x) = N(t,x)\mathrm{e}^{tS_d}$, one can prove that there exists a piecewise continuous solution N to (5.8).

The structure of the target system (5.7b)–(5.7d) directly implies that $\varepsilon(t,\cdot) = 0$ for $t \geq t_0 + \tilde{T}_{\text{unif}}(\Lambda)$, where $\tilde{T}_{\text{unif}}(\Lambda)$ is defined by

$$\tilde{T}_{\text{unif}}(\Lambda) = \sup_{t\geq 0}[s_2^{\text{out}}(s_1^{\text{out}}(t,0),1) - t]. \tag{5.10}$$

It follows from (5.9) that the observer gain function l_w is uniquely determined by l_d. In the next step, we extend the observer (5.4) to achieve a finite-time estimate for states v_d and w. To this end, let us notice that the function derived from the measurement and the observers

$$y_d(t) := y_m(t) - \hat{w}_2(t,0) + N_2(t,0)\hat{v}_d(t) \tag{5.11}$$

satisfies

$$y_d(t) = N_2(t,0)v_d(t), \quad \forall t \geq t_0 + \tilde{T}_{\text{unif}}(\Lambda). \tag{5.12}$$

Indeed, it follows from $\varepsilon(t,\cdot) = 0$ for $t \geq t_0+\tilde{T}_{\text{unif}}(\Lambda)$ that $e_{w,2}(t,0) = N_2(t,0)e_d(t)$ for $t \geq t_0+\tilde{T}_{\text{unif}}(\Lambda)$. By $e_d = v_d - \hat{v}_d$ and $e_w = w - \hat{w}$, we obtain that (5.12) holds. By using y_d, we consider the following finite-time disturbance observer, for (t,x) in $\mathscr{D}(t_0)$,

$$\dot{\hat{\mu}}_d(t) = S_d\hat{\mu}_d(t) + l_\mu(t)(y_d(t) - N_2(t,0)\hat{\mu}_d(t)), \tag{5.13a}$$

$$\dot{\hat{v}}_d(t) = S_d\hat{v}_d(t) + l_d(t)(y_m(t) - \hat{w}_2(t,0)), \tag{5.13b}$$

$$\partial_t\hat{w}(t,x) = \Lambda(t,x)\partial_x\hat{w}(t,x) + \hat{g}_1(t,x)\hat{v}_d(t) + l_w(t,x)(y_m(t) - \hat{w}_2(t,0)), \tag{5.13c}$$

$$\hat{w}_1(t,0) = q_0(t)y_m(t) + \hat{g}_2(t)^\top\hat{v}_d(t), \tag{5.13d}$$

$$\hat{w}_2(t,1) = q_1(t)\hat{w}_1(t,1) + u(t) + \hat{g}_3(t)\hat{v}_d(t), \tag{5.13e}$$

$$\hat{\mu}_d(t_0) = \hat{\mu}_d^0, \quad \hat{v}_d(t_0) = \hat{v}_d^0, \quad \hat{w}(t_0,x) = \hat{w}^0(x), \tag{5.13f}$$

with the additional initial condition $\hat{\mu}_d^0$. Therein, y_d is defined by (5.11) and the observer gain function l_μ is considered. By introducing the additional error $e_\mu = v_d - \hat{\mu}_d$ and noticing $\varepsilon(t,\cdot) = 0$ for $t \geq t_0 + \tilde{T}_{\mathrm{unif}}(\Lambda)$ and (5.12), we obtain that the error dynamics of observers $\hat{\mu}_d$ and $\hat{v}_d$ is given by, for $t \geq t_0 + \tilde{T}_{\mathrm{unif}}(\Lambda)$,

$$\dot{e}_\mu(t) = (S_d - l_\mu(t)N_2(t,0))e_\mu(t), \tag{5.14a}$$

$$\dot{e}_d(t) = (S_d - l_d(t)N_2(t,0))e_d(t). \tag{5.14b}$$

Let us use $(V(t), U(t))$ to represent the finite-dimensional linear time-varying control system

$$\dot{\nu}(t) = U(t)\nu(t), \quad \nu_y(t) = V(t)\nu(t),$$

where $\nu(t)$ in $\mathbb{R}^n$ is the state and $\nu_y(t)$ in $\mathbb{R}$ is the measurement. In order to derive a constructive way to compute suitable gain functions l_d and l_μ, we assume that the system $(N_2(t,0), S_d)$ is uniformly observable in the following sense (see Menold *et al.* (2003)).

Definition 2. The system $(V(t), U(t))$ is called uniformly observable if the observability matrix

$$\mathscr{Q}_{V,U}(t) = \begin{bmatrix} V(t) \\ L_U V(t) \\ \vdots \\ L_U^{n-1} V(t) \end{bmatrix}$$

has rank n for all times t, where the differential operator L_U is defined as $L_U V(t) := \dot{V}(t) + V(t)U(t)$.

While verifying the rank condition at a given time instant t is straightforward, ensuring its validity for all t proves considerably more challenging in practical applications. A possible approach is to verify this condition at all discrete time grid points, as implemented in Section 6. By the results in Bestle & Zeitz (1983); Menold *et al.* (2003), since $(N_2(t,0), S_d)$ is uniformly observable, there exists a transformation $T_{d,N}(t)$ that converts the system $(N_2(t,0), S_d)$ into the observer canonical form.

The inverse transformation matrix $T_{d,N}(t)^{-1}$ can be determined in columns

$$T_{d,N}(t)^{-1} = [q_{d,N}(t), \tilde{\mathscr{L}} q_{d,N}(t), \ldots, \tilde{\mathscr{L}}^{n_d-1} q_{d,N}(t)]$$

with the differential operator $\tilde{\mathscr{L}} q_{d,N}(t) = -\dot{q}_{d,N}(t) + S_d q_{d,N}(t)$, where $q_{d,N}(t)$ is defined by the last column of the inverse observability matrix

$$q_{d,N}(t) = \mathscr{Q}_{N_2(\cdot,0),S_d}(t)^{-1}[0, \ldots, 0, 1]^\top.$$

Denote $\bar{c} = [0, \ldots, 0, 1]$. By applying the transformation $T_{d,N}(t)$ to the error system (5.14), namely $\tilde{e}_\mu(t) = T_{d,N}(t) e_\mu(t)$ and $\tilde{e}_d(t) = T_{d,N}(t) e_d(t)$, we obtain that, for $t \geq t_0 + \tilde{T}_{\text{unif}}(\Lambda)$,

$$\dot{\tilde{e}}_\mu(t) = (E_{n_d} - (a_{d,N}(t) + T_{d,N}(t) l_\mu(t))\bar{c})\tilde{e}_\mu(t), \tag{5.15a}$$

$$\dot{\tilde{e}}_d(t) = (E_{n_d} - (a_{d,N}(t) + T_{d,N}(t) l_d(t))\bar{c})\tilde{e}_d(t), \tag{5.15b}$$

where

$$E_{n_d} = \begin{bmatrix} 0 & \ldots & \ldots & 0 \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 \end{bmatrix}, \quad a_{d,N}(t) = -T_{d,N}(t)\tilde{\mathscr{L}}^{n_d} q_{d,N}(t).$$

Since $(\bar{c}, E_{n_d})$ is observable, by Proposition 3, there exist $\tilde{l}_d$ and $\tilde{l}_\mu$ such that the matrix $F_d := \text{bdiag}(E_n - \tilde{l}_\mu \bar{c}, E_n - \tilde{l}_d \bar{c})$ is Hurwitz and for almost all $D_d > 0$, $\det \begin{bmatrix} B_{n_d} & e^{F_d D_d} B_{n_d} \end{bmatrix} \neq 0$. Moreover, the finite time observer $\hat{v}_d^+$ is given by

$$\hat{v}_d^+(t) = T_{d,N}(t)^{-1} \begin{bmatrix} \text{Id}_{n_d} & 0 \end{bmatrix} \begin{bmatrix} B_{n_d} & e^{F_d D_d} B_{n_d} \end{bmatrix}^{-1} [z(t) - e^{F_d D_d} z(t - D_d)], \tag{5.16}$$

where

$$z(t) = \begin{bmatrix} T_{d,N}(t)\hat{\mu}_d(t) \\ T_{d,N}(t)\hat{v}_d(t) \end{bmatrix} \tag{5.17}$$

with

$$l_\mu(t) = T_{d,N}(t)^{-1}(\tilde{l}_\mu - a_{d,N}(t)), \quad l_d(t) = T_{d,N}(t)^{-1}(\tilde{l}_d - a_{d,N}(t)). \tag{5.18}$$

Summarizing the results above, for the finite-time disturbance observer, we have the following result.

THEOREM 4. Let $D_d > 0$ be fixed. Let $\tilde{T}_{\text{unif}}(\Lambda)$ be given by (5.10). Let N is the solution to (5.8). Assume that $(N_2(t, 0), S_d)$ is uniformly observable. There exist $\tilde{l}_d$ and $\tilde{l}_\mu$ such that the finite time disturbance observer $\hat{v}_d^+$ is given by (5.16)–(5.18), and the finite time state observer $\hat{w}^+$ is given by

$$\hat{w}^+(t,x) = \hat{w}(t,x) + N(t,x)(\hat{v}_d^+(t) - \hat{v}_d(t)). \tag{5.19}$$

In (5.16), (5.17) and (5.19), $(\hat{\mu}_d, \hat{v}_d, \hat{w})$ is the solution to (5.13) with l_w given by (5.9), l_μ and l_d given by (5.18), and $\hat{\mu}_d(t) = \hat{\mu}_d^0$ and $\hat{v}_d(t) = \hat{v}_d^0$ for t in $[t_0 - D_d, t_0]$. More precisely, for any $t_0 \geq 0$, v_d^0, $\hat{v}_d^0$ and $\hat{\mu}_d^0$ in $\mathbb{R}^{n_d}$, and piecewise continuous w^0 and $\hat{w}^0$, we have $\hat{v}_d^+(t) = v_d(t)$ and $\hat{w}^+(t,\cdot) = w(t,\cdot)$ for $t \geq t_0 + \tilde{T}_{\mathrm{unif}}(\Lambda) + D_d$.

Proof. Since the error dynamics of the observers $\hat{\mu}_d$ and $\hat{v}_d$ is given by (5.14) for $t \geq t_0 + \tilde{T}_{\mathrm{unif}}(\Lambda)$. Then by the discussion above and the results in Menold *et al.* (2003), we have $\hat{v}_d^+(t) = v_d(t)$ for $t \geq t_0 + \tilde{T}_{\mathrm{unif}}(\Lambda) + D_d$.

Then let us prove $\hat{w}^+$ achieve the finite-time estimate for w. The structure of the error system (5.7b)–(5.7d) directly implies that $\varepsilon_w(t,\cdot) = 0$ for $t \geq t_0 + \tilde{T}_{\mathrm{unif}}(\Lambda)$. Then we have, for $t \geq t_0 + \tilde{T}_{\mathrm{unif}}(\Lambda)$ and x in $[0,1]$,

$$w(t,x) = \hat{w}(t,x) + N(t,x)(v_d(t) - \hat{v}_d(t)),$$

and therefore, for $t \geq t_0 + \tilde{T}_{\mathrm{unif}}(\Lambda)$ and x in $[0,1]$,

$$w(t,x) - \hat{w}^+(t,x) = N(t,x)(v_d(t) - \hat{v}_d^+(t)).$$

Consequently, $\hat{w}^+(t,\cdot) = w(t,\cdot)$ for $t \geq t_0 + \tilde{T}_{\mathrm{unif}}(\Lambda) + D_d$ follows from $\hat{v}_d^+(t) = v_d(t)$ for $t \geq t_0 + \tilde{T}_{\mathrm{unif}}(\Lambda) + D_d$. □

5.3. *Finite-time output feedback regulator*

In this section, let us combine the results of Theorem 2, Proposition 3 and Theorem 4 to obtain the finite-time output feedback regulator.

THEOREM 5. Assume that the assumptions of Theorem 2, Proposition 3 and Theorem 4 hold. Let $T_{\mathrm{unif}}(\Lambda)$ and $\tilde{T}_{\mathrm{unif}}(\Lambda)$ be given by (4.3) and (5.10) respectively. Let the finite-time observers $\hat{v}_r^+$, $\hat{v}_d^+$ and $\hat{w}^+$, and time delays D_r and D_d be given by Proposition 3 and Theorem 4 and let $\hat{v}^+ = (\hat{v}_d^{+\top}, \hat{v}_r^{+\top})^\top$. Then the output y of the system (2.1) achieves the uniform finite-time output regulation within settling time $T_{\min} := T_{\mathrm{unif}}(\Lambda) + \max\{D_r, \tilde{T}_{\mathrm{unif}}(\Lambda) + D_d\}$ by output feedback regulator. More precisely, there exists an output feedback regulator

$$u(t) = -q_1(t)\hat{w}_1^+(t,1) + k_v(t)^\top \hat{v}^+(t) \tag{5.20}$$

with feedback gain function k_v given by (4), such that for all $t_0 \geq 0$, v_r^0 in $\mathbb{R}^{n_r}$, $\hat{\underline{v}}_r^0$ in $\mathbb{R}^{2n_r}$, v_d^0, $\hat{v}_d^0$ and $\hat{\mu}_d^0$ in $\mathbb{R}^{n_d}$, and piecewise continuous w^0 and $\hat{w}^0$, the output tracking error e_y satisfies $e_y = 0$ for $t \geq t_0 + T_{\min}$.

Proof. It follows from Proposition 3 and Theorem 4 that $\hat{v}_r^+(t) = v_r(t)$, $\hat{v}_d^+(t) = v_d(t)$ and $\hat{w}^+(t,\cdot) = w(t,\cdot)$ for $t \geq t_0 + \max\{D_r, \tilde{T}_{\mathrm{unif}}(\Lambda) + D_d\}$. Then the proof is completed by Theorem 2. □

6. Numerical simulations

Consider the following transport equations, for $t \geq 0$, $0 \leq x \leq 1$,

$$\partial_t w_1(t,x) = -(1+0.5\sin(2\pi t))\partial_x w_1(t,x) + (1+0.1\sin(0.1t)\sin(0.05x))d_1(t), \tag{6.1a}$$
$$\partial_t w_2(t,x) = (1.5+\cos(2\pi t))\partial_x w_2(t,x) + (1+0.1\sin(0.05t)\sin(0.1x))d_1(t), \tag{6.1b}$$
$$w_1(t,0) = (1+0.5\cos(t))w_2(t,0) + (1+0.1\sin(0.2t))d_2(t), \tag{6.1c}$$
$$w_2(t,1) = w_1(t,1) + u(t) + (1+0.1\sin(0.3t))d_3(t), \tag{6.1d}$$
$$w(0,x) = w^0(x), \tag{6.1e}$$
$$y(t) = w_1(t,1) + (1+0.1\sin(0.4t))d_4(t), \tag{6.1f}$$
$$y_m(t) = w_2(t,0). \tag{6.1g}$$

It can be checked that Assumptions 1 to 4 hold so Theorem 2 applies. The signal model is given by (2.4) with the matrices

$$S_d = \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix}, \quad S_r = \begin{bmatrix} 0 & -0.8 \\ 0.8 & 0 \end{bmatrix} \tag{6.2}$$

and the vectors $p_1^\top = (1,0)$, $p_2^\top = (0,1)$, $p_3^\top = (-2,0)$, $p_4^\top = (0,3)$ and $p_r^\top = (1,0)$. By choosing $v_d(0)^\top = v_r(0)^\top = (1,0)$, we have $d_1(t) = \cos(0.5t)$, $d_2(t) = \sin(0.5t)$, $d_3(t) = -2\cos(0.5t)$, $d_4(t) = 3\sin(0.5t)$ and $r(t) = \cos(0.8t)$ for all $t \geq 0$. It is noted that the reference signal observer and the disturbance observer are independent of each other, and the reference signal observer has already been demonstrated in the numerical example of Deutscher (2017a). Therefore, we focus only on the disturbance observer here. The matrix N can be solved from (5.8) explicitly. The assumptions of Theorem 4, namely $(N_2(t,0), S_d)$ is uniformly observable, can be checked numerically, and therefore, Theorem 4 applies. Notice that the exit times for system (6.1) is $s_1^{\text{out}}(t,0) = t+1$ and $s_2^{\text{out}}(t,1)$ is the solution to $1.5(s_2^{\text{out}}(t,1)-t) + (\sin(2\pi s_2^{\text{out}}(t,1)) - \sin(2\pi t))/(2\pi) = 1$. After numerical calculations, $\max_{0\leq t\leq 10}[s_2^{\text{out}}(t,1)-t] \approx 0.7944$, and therefore, by applying Theorem 5, we have with (4.3) and (5.10) that the estimate of the settling time

$$T_{\text{unif}}(\varLambda) \approx 1.7944, \quad \tilde{T}_{\text{unif}}(\varLambda) \approx 1.7944. \tag{6.3}$$

We use the `place` function in MATLAB to calculate the observer gain functions l_d and l_μ. Here the `place` function is based on pole placement method. Specifically, we use the `place` function to select l_d (resp. l_μ) such that the eigenvalues of $S_d - l_d(t)N_2(t,0)$ (resp. $S_d - l_\mu(t)N_2(t,0)$) are $-1 \pm \mathrm{i}$ (resp. $-2 \pm \mathrm{i}$) for $0 \leq t \leq 10$. Then the observer gain function l_w follows from (5.9). We choose time delay $D_d = 1$. Then the feedback control u follows from (5.16), (5.19) and (5.20). For the initial conditions, we choose $\hat{v}_d^0 = \hat{\mu}_d^0 = 0$, $w^0(x) = 0$, $\hat{w}_1^0(x) = -1.8875x$ and $\hat{w}_2^0(x) = 0.1125x$ for $0 \leq x \leq 1$, which satisfy the zero-order compatibility conditions (see (Bastin & Coron, 2016, Chap. 4.5.2)).

To numerically check the finite-time output regulation, we use a 2-point upwind scheme (see (Allaire, 2005, Chap. 2.3.1), (Vande Wouwer *et al.*, 2014, Chap. 3)) to discrete (6.1) and the corresponding observer system. We select the parameters of the numerical scheme such that the Courant-Friedrichs-Lewy (CFL) condition for the stability holds. More precisely, we set the space discretization of $\varDelta x = 1/99$ and time discretization of $\varDelta t = 10/2999$.[1] The time evolutions of the output y, the

[1] Simulations have been performed in MATLAB. The simulation codes can be downloaded from Codes.

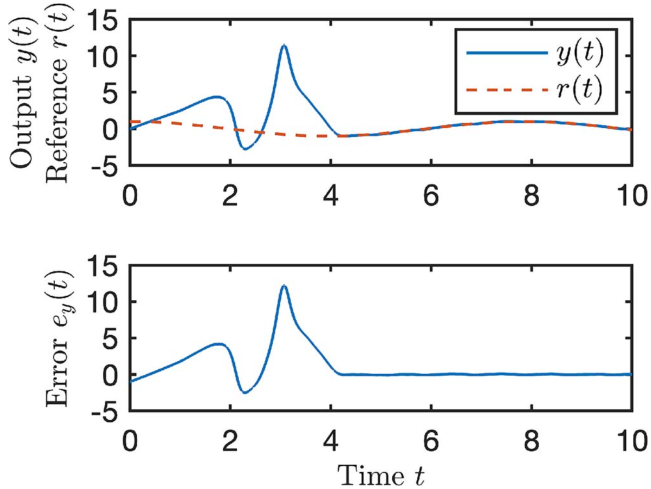

FIG. 1. Tracking behavior of (6.1) with output $y(t)$ for a reference signal $r(t) = \cos(0.8t)$. Top: time evolution of the output $y(t)$ (blue solid line) and of the reference signal $r(t)$ (red dashed line) for $0 \le t \le 10$. Down: time evolution of the output tracking error $e_y(t) = y(t) - r(t)$ for $0 \le t \le 10$.

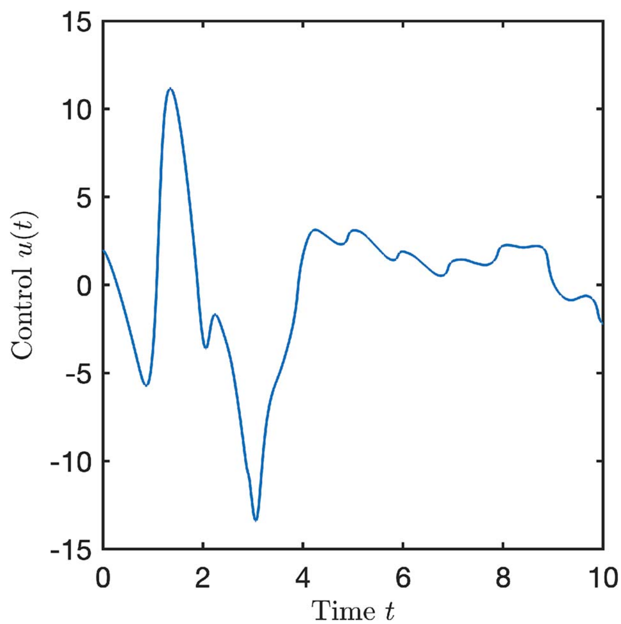

FIG. 2. Control input $u(t)$ for $0 \le t \le 10$.

reference signal r and the tracking error e_y are in Fig. 1. The control input $u(t)$ is in Fig. 2. The time evolutions of the disturbance v_d, the finite-time disturbance estimate $\hat{v}_d^+$ and the disturbance error $\hat{v}_d^+ - v_d$ are in Figs. 3 and 4. The solutions $w_1(t,x)$ and $w_2(t,x)$ to (6.1) are in Figs 5 and 6 respectively. The

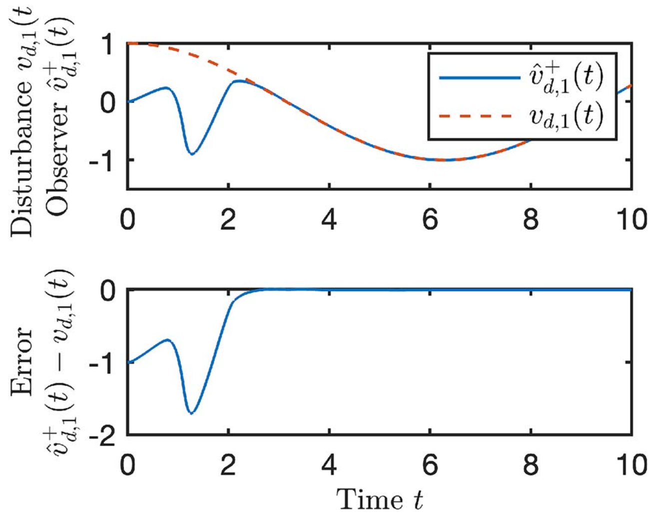

FIG. 3. Disturbance $v_{d,1}(t) = \cos(0.5t)$ and the finite-time disturbance estimate $\hat{v}^+_{d,1}(t)$ of $v_{d,1}(t)$. Top: time evolution of the disturbance $v_{d,1}(t)$ (red dashed line) and of the finite-time estimate $\hat{v}^+_{d,1}(t)$ (blue solid line) for $0 \le t \le 10$. Down: time evolution of the disturbance error $\hat{v}^+_{d,1}(t) - v_{d,1}(t)$ for $0 \le t \le 10$.

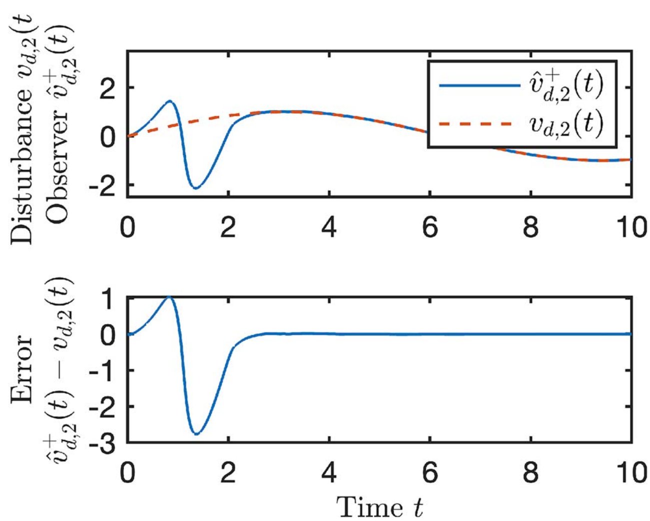

FIG. 4. Disturbance $v_{d,2}(t) = \sin(0.5t)$ and the finite-time disturbance estimate $\hat{v}^+_{d,2}(t)$ of $v_{d,2}(t)$. Top: time evolution of the disturbance $v_{d,2}(t)$ (red dashed line) and of the finite-time estimate $\hat{v}^+_{d,2}(t)$ (blue solid line) for $0 \le t \le 10$. Down: time evolution of the disturbance error $\hat{v}^+_{d,2}(t) - v_{d,2}(t)$ for $0 \le t \le 10$.

finite-time state estimate $\hat{w}^+_1(t,x)$ and $\hat{w}^+_2(t,x)$ are in Figs. 7 and 8 respectively. We can observe that for $t \ge T_{\text{unif}}(\Lambda) + \tilde{T}_{\text{unif}}(\Lambda) + D_d$ with $T_{\text{unif}}(\Lambda)$ and $\tilde{T}_{\text{unif}}(\Lambda)$ given by (6.3), the tracking error vanishes. This verifies finite-time output regulation with settling time $T_{\text{unif}}(\Lambda) + \tilde{T}_{\text{unif}}(\Lambda) + D_d$ and therefore,

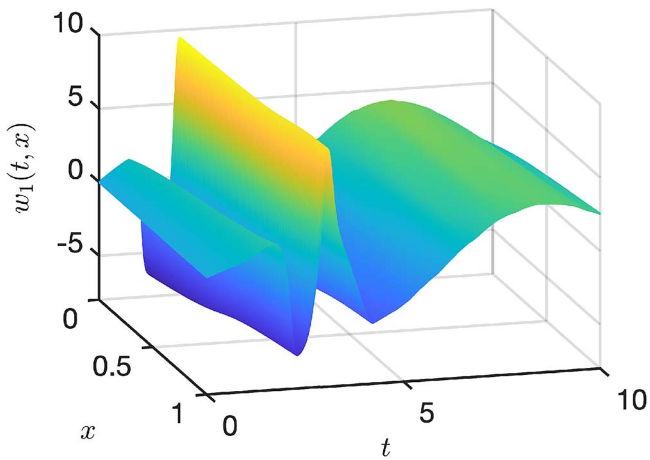

FIG. 5. Solution $w_1(t,x)$ to (6.1) for $0 \le t \le 10$ and for $0 \le x \le 1$.

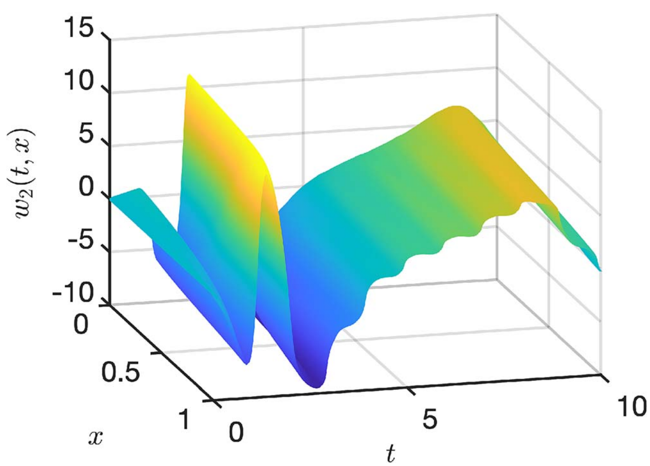

FIG. 6. Solution $w_2(t,x)$ to (6.1) for $0 \le t \le 10$ and for $0 \le x \le 1$.

Theorem 5 is verified. Notice that there is a small fluctuation in the tracking error $e_y(t)$ for $t \ge T_{\text{unif}}(\Lambda)$. The reason lies in the truncation error of the numerical scheme which is $\mathscr{O}(\Delta t + \Delta x)$ (see (Allaire, 2005, Chap. 2.3.1)).

7. Concluding remarks

This work addresses the finite-time output regulation for linear time-varying hyperbolic system by state feedback regulator and output feedback regulator. For the finite-time disturbance observer design, more regularities of the system parameters are needed. A natural question is whether the assumption that $(N_2(t,0), S_d)$ is uniformly observable can be described by a more explicit condition, that is, the relationship between the disturbance model and the hyperbolic system. In Deutscher (2017a), this

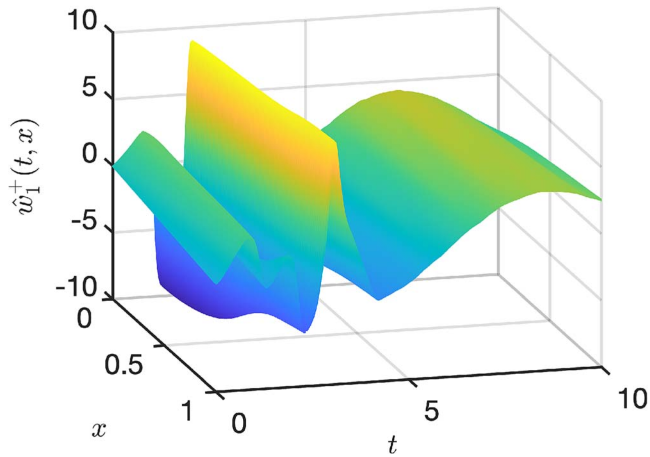

FIG. 7. Finite-time estimate $\hat{w}_1^+(t,x)$ of $w_1(t,x)$ for $0 \leq t \leq 10$ and for $0 \leq x \leq 1$.

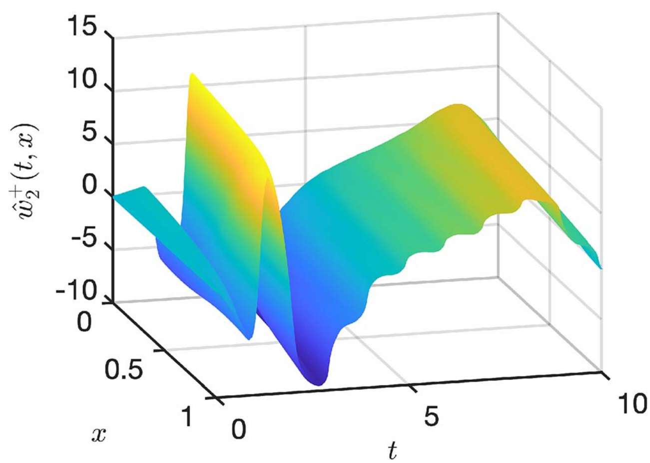

FIG. 8. Finite-time estimate $\hat{w}_2^+(t,x)$ of $w_2(t,x)$ for $0 \leq t \leq 10$ and for $0 \leq x \leq 1$.

condition was explicitly expressed for time-invariant systems. However, in time-varying systems, this is a challenging problem.

Data availability

The data underlying this article are available in the article and in its online supplementary material.

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