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Exact controllability for a Rayleigh beam with piezoelectric actuator

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ABSTRACT

In this paper, exact controllability problem for a Rayleigh beam with piezoelectric actuator is considered. Controllability results show that the space of controllable initial data depends on the regularity of the control and the location of the actuator. Two different spaces of control, $L^2(0,T)$ and $(H^1(0,T))'$, correspond to two different controllability properties, L^2-controllability and $(H^1)'$-controllability, respectively. The approach to prove controllability results is based on the Hilbert Uniqueness Method. Some non-controllability results are also obtained. In particular, non-controllability in short control time is studied by using Riesz basis property of exponential family in $L^2(0,T)$. Finally, minimal control time for exact controllability is obtained.

1. Introduction and main results

1.1. History

In recent decades, there have been a large number of papers concerning the study of flexible structures. Three main directions of research can be considered: the modelling problem, the controllability problem and the stabilization problem. Modelling a flexible structure as a beam equation or a plate equation is an essential research field. In [1], four types of models for the transversely vibrating uniform beam, i.e., Euler–Bernoulli beam, Rayleigh beam, shear beam and Timoshenko beam, were summarized and analysed. In the past few decades, the study of elastic structures with a piezoelectric actuator or sensor has gained a lot of attention. See [2,3] for a PDE modelling elastic structures with a piezoelectric actuator or sensor.

Concerning controllability for PDEs, boundary controllability for wave equation and plate equation was studied in [4] using the Hilbert Uniqueness Method (HUM). There were plenty of works on controllability for beam and plate based on HUM. In [4,5], boundary controllability for Kirchhoff plate equation was fully investigated. Exact controllability was obtained in sufficient large control time with a single boundary control (active on a sufficiently large portion of the boundary) in the case of clamped boundary conditions.

As for control problems of beam equation, exact controllability for Euler–Bernoulli beam hinged at both ends with piezoelectric actuator was firstly considered in [6]. Since the space dimension is one, Fourier series was used in [6]. Then [7] studied the exact controllability for the same beam equation with piezoelectric actuator in a different physical configuration: the clamped-free boundary conditions, i.e. a beam clamped at one end and free at the other end. Besides, the stabilization problem for the same equation was considered in [8]. In our previous work [9], the stabilization of the Rayleigh beam equation with piezoelectric actuator was investigated. Polynomial decay rate was obtained by an output feedback. In order to handle the piezoelectric actuator, some results from the theory of Diophantine approximation are used in [6–9] and in the present work. The controllability or the stabilization results in these papers are highly relevant to the position of the piezoelectric actuator. In [10], Ingham inequality (see [11, 12]) was used to obtain exact controllability for Rayleigh beam equation with a single boundary control among four different boundary conditions.

In addition, there are many advances in the control literature for smart-material structures. As for controllability problem, [13] studied the time optimal control, for a Kirchhoff plate equation with distributed control. The main contribution of [13] is that the time optimal control is proved to satisfy the bang–bang property. In this paper, the optimal control time is also obtained. For the control problem of piezoelectric beam, [14,15] investigated the well-posedness, stabilization and exact observability of voltage-actuated piezoelectric beams with magnetic effects. In [16,17], well-posedness and stabilization of current-controlled piezoelectric beams was considered. Concerning the semidiscretized approximation for exact observability and controllability of beam equation, [18,19] studied approximation of Euler–Bernoulli beam control system with clamped boundaries and clamped-free boundaries respectively.

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1.2. Problem statement

In this paper, we consider the control problem modelling the transverse deflection of a Rayleigh beam which is subject to the action of an attached piezoelectric actuator. Assuming that the beam is hinged at both ends, the equation of Rayleigh beam can be written as (see, for instance, [20] where the equations are explicitly derived from [2,3]), for (x,t) in $(0,\pi)\times(0,+\infty)$,

$$w_{tt}(x,t)-\alpha w_{xxtt}(x,t)+w_{xxxx}(x,t)=u(t)\frac{\mathrm{d}}{\mathrm{d}x}[\delta_\eta(x)-\delta_\xi(x)], \tag{1.1a}$$

$$w(0,t)=w(\pi,t)=w_{xx}(0,t)=w_{xx}(\pi,t)=0, \tag{1.1b}$$

$$w(x,0)=w^0(x),\ w_t(x,0)=w^1(x). \tag{1.1c}$$

In the equations above w represents the transverse deflection of the beam, $\alpha>0$ is a physical constant, ξ and η stand for the ends of the actuator ($0<\xi<\eta<\pi$), and δ_y is the Dirac mass at the point y. The control is given by the function $u:[0,T]\to\mathbb{R}$ standing for the time variation of the voltage applied to the actuator.

Our main purpose is to find the initial data that can be steered to rest by means of the control u. To give the precise definitions of exact controllability, let us introduce for any ω in $\mathbb{R}$ the functional space Y_ω as follows. Let $Y_0=L^2(0,\pi)$. For $\omega>0$, let Y_ω be the closure in $H^\omega(0,\pi)$ of the set of y in $C^\infty([0,\pi])$ satisfying the conditions

$$y^{(2n)}(0)=y^{(2n)}(\pi)=0\quad\forall n\geq 0. \tag{1.2}$$

For $\omega<0$, let Y_ω be the dual space of $Y_{-\omega}$ with respect to the space Y_0. Then we give the precise definitions.

Definition 1.1. The initial data (w^0,w^1) in $Y_2\times Y_1$ is exactly L^2-controllable in (ξ,η) at time T if there exists u in $L^2(0,T)$ such that the solution w of (1.1) satisfies the condition $w(\cdot,T)=w_t(\cdot,T)=0$.

Definition 1.2. The initial data (w^0,w^1) in $Y_1\times Y_0$ is exactly $(H^1)'$-controllable in (ξ,η) at time T if there exists u in $(H^1(0,T))'$ such that the solution w of (1.1) satisfies the condition $w(\cdot,T)=w_t(\cdot,T)=0$.

Note that in Definitions 1.1 and 1.2, the spaces where the initial data (w^0,w^1) can be taken depend on the well-posedness of (1.1) (see Section 3). In Definition 1.2, the space $(H^1(0,T))'$ is the dual space of $H^1(0,T)$ with respect to $L^2(0,T)$. The study of controllability with less regular control is inspired by [4] which studied the controllability of changing the norm for wave equation and plate equation. Note that the system (1.1) is a time-reversible linear system, so exact controllability is equivalent to null controllability (see Theorem 2.41 of [21, p. 55]).

1.3. Contributions of the paper

In this work, the exact controllability problem is solved for Rayleigh beam equation given in [20]. To do that, weak control, namely $(H^1(0,T))'$ control is considered. To this end, a new well-posedess result is needed and proved. In addition to this controllability result, several non-controllability results are provided in this paper. The first one concerns the location of the piezoelectric actuator and describes the set of actuator ends so that controllability holds or does not hold. The second one solves the non-controllability problem for less regular initial data. The third one is about the lack of controllability in short control time. Especially, the third non-controllability result, together with the exact controllability results, reveals the minimal control time for this problem. To the best of our knowledge, this is the first result concerning the minimal control time of the exact controllability for Rayleigh beam.

1.4. Controllability results

In order to state the exact controllability results, let $\varepsilon>0$ and let the sets $A\subset(0,1)$ and $B_\varepsilon\subset(0,1)$ be the sets defined in Section 2. From Section 2, the set A is uncountable and has zero Lebesgue measure, and the Lebesgue measure of set B_ε is 1.

Let us first recall a result in the conference paper [22]:

Theorem 1.3 (*[22]*). *Let $T>2\pi\sqrt{\alpha}$ and $\varepsilon>0$.*

1. *Suppose that $\frac{\eta+\xi}{2\pi}$ and $\frac{\eta-\xi}{2\pi}$ belong to the set A. Then all initial data in $Y_4\times Y_3$ are exactly L^2-controllable in (ξ,η) at time T.*
2. *Suppose that $\frac{\eta+\xi}{2\pi}$ and $\frac{\eta-\xi}{2\pi}$ belong to the set B_ε. Then all initial data in $Y_{4+2\varepsilon}\times Y_{3+2\varepsilon}$ are exactly L^2-controllable in (ξ,η) at time T.*

Theorem 1.3 states two L^2-controllability results. The first result shows that, for the end of the piezoelectric actuator in an uncountable zero measure set, we have L^2-controllability in $Y_4\times Y_3$. The second result implies that, for almost all choices of the end of the piezoelectric actuator, we have L^2-controllability in more regular Sobolev spaces than $Y_4\times Y_3$.

Our exact controllability results are introduced in the following theorem which concerns the exact controllability in less regular spaces.

Theorem 1.4. *Let $T>2\pi\sqrt{\alpha}$ and $\varepsilon>0$.*

1. *Suppose that $\frac{\eta+\xi}{2\pi}$ and $\frac{\eta-\xi}{2\pi}$ belong to the set A. Then all initial data in $Y_3\times Y_2$ are exactly $(H^1)'$-controllable in (ξ,η) at time T.*
2. *Suppose that $\frac{\eta+\xi}{2\pi}$ and $\frac{\eta-\xi}{2\pi}$ belong to the set B_ε. Then all initial data in $Y_{3+2\varepsilon}\times Y_{2+2\varepsilon}$ are exactly $(H^1)'$-controllable in (ξ,η) at time T.*

In Theorem 1.4, $(H^1(0,T))'$ control brings new difficulties to the problem. The well-posedness of (1.1) with u in $(H^1(0,T))'$ needs to be proven while the well-posedness of (1.1) with u in $L^2(0,T)$ is a known result (see Section 3).

Theorems 1.3 and 1.4 give some sufficient conditions for exact controllability. All the results show the dependence of the space of exactly controllable initial data on the location of actuator. The differences between these two theorems are the different spaces of the control and the different spaces of the controllable initial data. In Theorem 1.4, the control belongs to $(H^1(0,T))'$ rather than $L^2(0,T)$ and the space of the controllable initial data is larger than the space in Theorem 1.3 with the same choice of ξ and η. Roughly speaking, the larger (less regular) the space of control is, the larger (less regular) the space of controllable initial data is. To the best knowledge of the authors, such a result has not been developed yet for beam equation with piezoelectric actuator or internal control.

1.5. Non-controllability results

After the controllability results given in the previous subsection, let us state some non-controllability results. In Section 4, from Propositions 4.1 and 4.3 and the solution (3.5) of the adjoint problem, we can see that condition

$$\frac{\eta-\xi}{2\pi},\ \frac{\eta+\xi}{2\pi}\in\mathbb{R}\setminus\mathbb{Q} \tag{1.3}$$

is necessary to have any exact controllability result. The first non-controllability result concerns the insufficiency of condition (1.3). Under condition (1.3), there exist ξ, η and initial condition of problem (1.1) such that no control can steer this initial condition to the equilibrium.

Theorem 1.5.

1. *For any $\beta\geq-1$, there exist ξ and η satisfying (1.3) such that for any $T>0$, the space $Y_{\beta+3}\times Y_{\beta+2}$ contains some initial data that are not exactly L^2-controllable in (ξ,η) at time T.*

2. *For any $\beta \geq -2$, there exist ξ and η satisfying (1.3) such that for any $T > 0$, the space $Y_{\beta+3} \times Y_{\beta+2}$ contains some initial data that are not exactly $(H^1)'$-controllable in (ξ, η) at time T.*

Theorem 1.3 (resp. Theorem 1.4) gives no information on L^2-controllability (resp. $(H^1)'$-controllability) of initial data in $Y_{\beta+3} \times Y_{\beta+2}$ for $\beta < 1$ (resp. for $\beta < 0$). A partial answer is given by the following result.

Theorem 1.6. *Let $\varepsilon > 0$, $T > 0$ and ξ, η in $(0, \pi)$ be arbitrary.*

1. *The set $Y_{3-\varepsilon} \times Y_{2-\varepsilon}$ contains some initial data that are not exactly L^2-controllable in (ξ, η) at time T.*
2. *The set $Y_{2-\varepsilon} \times Y_{1-\varepsilon}$ contains some initial data that are not exactly $(H^1)'$-controllable in (ξ, η) at time T.*

Notice that all the exact controllability results in Theorems 1.3 and 1.4 require $T > 2\pi\sqrt{\alpha}$, however, in [6], the exact controllability results for Euler–Bernoulli beam have no requirement of control time. Consequently, a huge difference between Rayleigh beam and Euler–Bernoulli beam is revealed, and the reason lies in various distributions of their eigenvalues. More precisely, under the same boundary condition (1.1b), the eigenvalues of Rayleigh beam equation are $\frac{k^4}{1+\alpha k^2}$ for k in $\mathbb{N}^*$ (see Section 3.1) while the eigenvalues of Euler–Bernoulli beam equation are k^4 for k in $\mathbb{N}^*$ (see [6]). Roughly speaking, this fact implies that Rayleigh beam equation possesses finite propagation speed and that Euler–Bernoulli beam equation possesses infinite propagation speed. For this reason, all the exact controllability results for Rayleigh beam require $T > 2\pi\sqrt{\alpha}$ while the exact controllability results for Euler–Bernoulli beam hold for all $T > 0$ (see [6]). Based on this fact, we give the non-controllability results for $0 < T < 2\pi\sqrt{\alpha}$.

Theorem 1.7. *Let $0 < T < 2\pi\sqrt{\alpha}$ and ξ, η in $(0, \pi)$ be arbitrary.*

1. *For any $\beta \geq -1$, the space $Y_{\beta+3} \times Y_{\beta+2}$ contains initial data that are not exactly L^2-controllable in (ξ, η) at time T.*
2. *For any $\beta \geq -2$, the space $Y_{\beta+3} \times Y_{\beta+2}$ contains initial data that are not exactly $(H^1)'$-controllable in (ξ, η) at time T.*

Remark 1.8. For the case $T = 2\pi\sqrt{\alpha}$, whether the exact controllability still holds remains open.

Notice that in Theorem 1.5, the lack of controllability holds for some special ξ and η which are related to the space of initial data. However, in Theorems 1.6 and 1.7, non-controllability holds for any ξ and η. The first part of this theorem (non-controllability with L^2 control function) was presented in the conference paper [22]. From Theorem 1.7, we can see that $T \geq 2\pi\sqrt{\alpha}$ is necessary for exact controllability for Rayleigh beam equation. Therefore, minimal control time for exact controllability is obtained. As far as we know, this is the first result stating a lack of L^2-controllability as well as $(H^1)'$-controllability for Rayleigh beam in short control time.

The remaining part of this paper is organized as follows. In Section 2, we provide some preliminaries on the theory of Diophantine approximation and the Riesz basis property of exponential family. The well-posedness results for the control problem (1.1) are given in Section 3. The main results are proved in Section 4. Appendix provides the proof of a technical lemma which is used in the proof of non-controllability in short control time.

2. Preliminaries

In this section, we provide some known results on the theory of Diophantine approximation (see [23,24]) and the Riesz basis property of exponential family (see [25]).

For a real number ρ, we denote by $\|\rho\|_{\mathbb{Z}}$ the difference, taken positively, between ρ and the nearest integer, i.e., $\|\rho\|_{\mathbb{Z}} = \min_{n\in\mathbb{Z}} |\rho - n|$. Let us denote by A the set of all irrationals ρ in $(0, 1)$ such that if $[0, a_1, \ldots, a_n \ldots]$ is the expansion of ρ as a continued fraction, then (a_n) is bounded. The set A is uncountable, and its Lebesgue measure is equal to zero (see Theorem I of [23, p. 120]). The following property proven in Theorem 6 of [24, p. 23] is essential for this paper.

Proposition 2.1. *A number ρ is in A if and only if there exists a constant $C > 0$ such that*

$$\|q\rho\|_{\mathbb{Z}} \geq \frac{C}{q} \tag{2.1}$$

for any strictly positive integer q.

The next proposition, which is proved in [23, p. 120], shows that an inequality slightly weaker than (2.1) holds for almost all points in $(0, 1)$.

Proposition 2.2. *For any $\varepsilon > 0$, there exists a set $B_\varepsilon \subset (0, 1)$ having Lebesgue measure equal to 1 such that for any ρ in B_ε, there exists a constant $C > 0$ such that for any strictly positive integer q, we have*

$$\|q\rho\|_{\mathbb{Z}} \geq \frac{C}{q^{1+\varepsilon}}. \tag{2.2}$$

The following proposition on simultaneous approximation proven in Theorem VII of [23, p. 14] is useful to prove Theorem 1.6.

Proposition 2.3. *Let $\rho_1, \ldots, \rho_k$ be k irrationals in $(0, 1)$. Then there exists a strictly increasing sequence of natural numbers q_n such that for all $n \geq 1$,*

$$q_n^{\frac{1}{k}} \max_{i=1,\ldots,k} (\|q_n\rho_1\|_{\mathbb{Z}}, \ldots, \|q_n\rho_i\|_{\mathbb{Z}}, \ldots, \|q_n\rho_k\|_{\mathbb{Z}}) \leq \frac{k}{k+1}.$$

The next proposition proven in Theorem II.4.18 of [25, p. 109] on the Riesz basis property of exponential family in $L^2(0, T)$ is essential to prove Theorem 1.7.

Proposition 2.4. *Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of complex numbers such that $\sup_{n\in\mathbb{Z}} |\mathrm{Im}\lambda_n| < \infty$ and $\inf_{n\neq m} |\lambda_m - \lambda_n| > 0$. Let $N(x, r) := \sharp\{\lambda_n | x \leq \mathrm{Re}\lambda_n < x + r\}$ for x in $\mathbb{R}$ and $r > 0$, where $\sharp\mathcal{A}$ is the number of elements in the set $\mathcal{A}$. Assume that for some $T > 0$,*

$$\lim_{r\to\infty} \frac{N(x, r)}{r} = \frac{T}{2\pi}$$

holds uniformly relative to all x in $\mathbb{R}$. Then for any T' in $(0, T)$, $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ contains a subfamily $\{e^{i\lambda_{q_n} t}\}_{n\in\mathbb{Z}}$ that forms a Riesz basis in $L^2(0, T')$. Moreover, if $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a sequence of real numbers such that $\lambda_n = -\lambda_{-n}$, the subsequence $\{\lambda_{q_n}\}_{n\in\mathbb{Z}}$ satisfies $\lambda_{q_n} = -\lambda_{q_{-n}}$.

3. Well-posedness of (1.1)

In Section 3.1, we show the well-posedness result of system (1.1) with u in $L^2(0, T)$ which has been proved in [20]. In Section 3.2, we prove the well-posedness and regularity results of system (1.1) with u in $(H^1(0, T))'$. The approach is inspired by [4].

3.1. Well-posedness of (1.1) with control in $L^2(0, T)$

In this section, let us recall two results in [20]. The first one provided the well-posedness of (1.1) with u in $L^2(0, T)$. The second one gave the well-posedness of the adjoint problem of (1.1) and some trace regularities. We keep the proof of the second one since some part of the proof is also used in other sections.

The following theorem is the well-posedness result of (1.1) with u in $L^2(0, T)$.

Theorem 3.1. *Suppose that (w^0, w^1) belongs to $Y_2 \times Y_1$. For any u in $L^2(0, T)$ and for any ξ and η in $(0, \pi)$, the initial and boundary value problem (1.1) admits a unique solution w having the regularity*

$$w \in C([0, T]; Y_2) \cap C^1([0, T]; Y_1). \tag{3.1}$$

Then let us consider the adjoint problem of (1.1) in $(0,\pi)\times(0,+\infty)$,

$$\phi_{tt}(x,t)-\alpha\phi_{xxtt}(x,t)+\phi_{xxxx}(x,t)=0, \tag{3.2a}$$

$$\phi(0,t)=\phi(\pi,t)=\phi_{xx}(0,t)=\phi_{xx}(\pi,t)=0, \tag{3.2b}$$

$$\phi(x,0)=\phi^0(x),\ \phi_t(x,0)=\phi^1(x). \tag{3.2c}$$

The next lemma provides the well-posedness of the adjoint problem (3.2) and some trace regularities needed in the proof of the main results.

Lemma 3.2. *For any initial data (ϕ^0,ϕ^1) in $Y_2\times Y_1$, there exists a unique weak solution ϕ of (3.2) in the class $C([0,T];Y_2)\cap C^1([0,T];Y_1)$. Moreover, for any χ in $(0,\pi)$, $\phi_x(\chi,\cdot)$ belongs to $H^1(0,T)$, and there exist $C,C'>0$ such that*

$$\|\phi_x(\chi,\cdot)\|^2_{H^1(0,T)}\le C(\|\phi^0\|^2_{H^2(0,\pi)}+\|\phi^1\|^2_{H^1(0,\pi)}), \tag{3.3}$$

$$\|\phi_x(\chi,\cdot)\|^2_{L^2(0,T)}\le C'(\|\phi^0\|^2_{H^1(0,\pi)}+\|\phi^1\|^2_{L^2(0,\pi)}). \tag{3.4}$$

Proof. It is easy to see, by the semigroup method, that the problem (3.2) admits a unique solution ϕ in $C([0,T];Y_2)\cap C^1([0,T];Y_1)$ (see [26, p. 104]).

Next we prove (3.3) and (3.4). Since the family of functions $\{x\mapsto\sin(kx)\}_{k\in\mathbb{N}^*}$ is the orthogonal basis of Y_1 and Y_2 respectively, let $\phi^0(x)=\sum_{k\ge1}a_k\sin(kx)$ and $\phi^1(x)=\sum_{k\ge1}b_k\sin(kx)$ with (k^2a_k) and (kb_k) in $l^2(\mathbb{R})$. By standard computation, we have

$$\phi(x,t)=\sum_{k\ge1}\left[a_k\cos\left(\frac{k^2}{\sqrt{1+\alpha k^2}}t\right)+\frac{b_k\sqrt{1+\alpha k^2}}{k^2}\sin\left(\frac{k^2}{\sqrt{1+\alpha k^2}}t\right)\right]\sin(kx). \tag{3.5}$$

Then for all $T>0$, $\phi_x(\chi,\cdot)$ belongs to $H^1(0,T)$ and

$$\int_0^T|\phi_{xt}(\chi,t)|^2\mathrm{d}x\le C\sum_{k\ge1}k^2(a_k^2k^2+b_k^2),$$

which yields (3.3). And simultaneously we have

$$\int_0^T|\phi_x(\chi,t)|^2\mathrm{d}x\le C'\sum_{k\ge1}(a_k^2k^2+b_k^2),$$

which clearly yields (3.4). □

3.2. Well-posedness of (1.1) with control in $(H^1(0,T))'$

As the control u belongs to $(H^1(0,T))'$, the dual space of $H^1(0,T)$, we need to define the solution of (1.1) in the weak form. The next proposition is the Riesz Representation Theorem for $H^1(0,T)$ (see [27, p. 62]).

Proposition 3.3. *For every u in $(H^1(0,T))'$, there exist functions u_0 and u_1 in $L^2(0,T)$ such that for all ϕ in $H^1(0,T)$,*

$$\langle u,\phi\rangle_{(H^1(0,T))'\times H^1(0,T)}=\int_0^T(u_0\phi+u_1\phi_t)\mathrm{d}t. \tag{3.6}$$

Note that u_0 and u_1 also define a distribution $\widetilde{u}$ in $\mathcal{D}'(0,T)$ as $\widetilde{u}=u_0-u_{1,t}$. We know from [27, p. 63] that the element u of $(H^1(0,T))'$ is an extension to $H^1(0,T)$ of the distribution $\widetilde{u}$.

Inspired by [4], we define the weak solution of (1.1) by transposition and prove the well-posedness. We explain the results in three steps.

1. Prove the well-posedness and the trace regularity of a non-homogeneous problem (3.8).
2. Define the weak solution of (1.1) by transposition.
3. Prove the well-posedness of (1.1).

Before we start, let us introduce two linear operators

$$\mathcal{L}:=I-\alpha\partial_{xx},\quad \mathcal{R}:=(I-\alpha\partial_{xx})^{-1}. \tag{3.7}$$

It follows from Lax–Milgram Theorem that the operator $\mathcal{L}$ (resp. $\mathcal{R}$) is an isomorphism from Y_i to Y_{i-2} (resp. Y_{i-2} to Y_i), $i=0,1,2$.

Step 1. Let $\{f,\theta^0,\theta^1\}$ belong to $L^1(0,T;Y_{-1})\times Y_2\times Y_1$. Let us consider the following backward non-homogeneous problem in $(0,\pi)\times(0,T)$,

$$\theta_{tt}(x,t)-\alpha\theta_{xxtt}(x,t)+\theta_{xxxx}(x,t)=f(x,t), \tag{3.8a}$$

$$\theta(0,t)=\theta(\pi,t)=\theta_{xx}(0,t)=\theta_{xx}(\pi,t)=0, \tag{3.8b}$$

$$\theta(x,T)=\theta^0,\ \theta_t(x,T)=\theta^1. \tag{3.8c}$$

The following proposition provides the well-posedness and the trace regularity of (3.8). Our approach to prove this proposition is inspired by methods used in [20,28].

Proposition 3.4. *For any initial data (θ^0,θ^1) in $Y_2\times Y_1$ and f in $L^1(0,T;Y_{-1})$, there exists a unique weak solution θ of (3.8) in the class $C([0,T];Y_2)\cap C^1([0,T];Y_1)$. Moreover, for any χ in $(0,\pi)$, $\theta_x(\chi,\cdot)$ belongs to $H^1(0,T)$, and there exists $C>0$ such that*

$$\|\theta_x(\chi,\cdot)\|_{H^1(0,T)}\le C(\|\theta^0\|_{H^2(0,\pi)}+\|\theta^1\|_{H^1(0,\pi)}+\|f\|_{L^1(0,T;Y_{-1})}). \tag{3.9}$$

Proof. Applying $\mathcal{R}$ to both sides of (3.8a), we obtain

$$\theta_{tt}(x,t)+\mathcal{R}\theta_{xxxx}(x,t)=\mathcal{R}f(x,t). \tag{3.10}$$

Notice that $\mathcal{R}f$ is in $L^1(0,T;Y_1)$. Then the problem (3.10) admits a unique solution θ in $C([0,T];Y_2)\cap C^1([0,T];Y_1)$ by the classical semigroup method (see [26, p. 106]). Moreover, there exists a constant $C_T>0$ such that

$$\|\theta\|_{C([0,T];Y_2)}\le C_T(\|\theta^0\|_{H^2(0,\pi)}+\|\theta^1\|_{H^1(0,\pi)}+\|f\|_{L^1(0,T;Y_{-1})}). \tag{3.11}$$

Then we need to prove inequality (3.9). The following lemma proved in [20,28] shows that the operator $\mathcal{R}\partial_{xxxx}$ is "similar" to the elliptic operator $-\frac{1}{\alpha}\partial_{xx}$.

Lemma 3.5. *The linear operator $L=-\frac{1}{\alpha}\partial_{xx}-\mathcal{R}\partial_{xxxx}$ is bounded from Y_2 to Y_2.*

Using this lemma, we can reduce the proof of (3.9) to a regularity property for a string equation. We consider the initial value problem in $(0,\pi)\times(0,T)$,

$$\theta_{1,tt}(x,t)-\frac{1}{\alpha}\theta_{1,xx}(x,t)=\mathcal{R}f(x,t),$$
$$\theta_1(0,t)=\theta_1(\pi,t)=0,$$
$$\theta_1(x,T)=\theta^0,\ \theta_{1,t}(x,T)=\theta^1.$$

The relations above imply that in $(0,\pi)\times(0,T)$, $\theta_2=\theta-\theta_1$ satisfies

$$\theta_{2,tt}(x,t)-\frac{1}{\alpha}\theta_{2,xx}(x,t)=L\theta,$$
$$\theta_2(0,t)=\theta_2(\pi,t)=0,$$
$$\theta_2(x,T)=0,\ \theta_{2,t}(x,T)=0.$$

Since θ belongs to $C([0,T];Y_2)$ and L is bounded from Y_2 to Y_2, $L\theta$ belongs to $C([0,T];Y_2)$. Then by the classical theory for evolution equations of hyperbolic type (see [29]), θ_2 belongs to $C([0,T];H^3(0,\pi))\cap C^1([0,T];H^2(0,\pi))$, and there exists a constant $C_T^1>0$ such that

$$\|(\theta_2,\theta_{2,t})\|_{C([0,T];H^3(0,\pi)\times H^2(0,\pi))}\le C_T^1\|\theta\|_{C([0,T];Y_2)}.$$

This inequality, combined with (3.11) and the standard trace theorem of hyperbolic equation, implies that there exists a constant $C_T^2>0$ such that for any χ in $(0,\pi)$,

$$\|\theta_{2,x}(\chi,\cdot)\|_{H^1(0,T)}\le C_T^2(\|\theta^0\|_{H^2(0,\pi)}+\|\theta^1\|_{H^1(0,\pi)}+\|f\|_{L^1(0,T;Y_{-1})}). \tag{3.12}$$

As for θ_1, it is already proved in [30] that there exists a constant $C_T^3 > 0$ such that for any χ in $(0,\pi)$,

$$\|\theta_{1,x}(\chi,\cdot)\|_{H^1(0,T)} \le C_T^3(\|\theta^0\|_{H^2(0,\pi)} + \|\theta^1\|_{H^1(0,\pi)} + \|f\|_{L^1(0,T;Y_{-1})}). \quad (3.13)$$

Then inequality (3.9) follows from (3.12) and (3.13). □

Step 2. Now we give the definition of the weak solution of (1.1). Denote by $\mathcal{X}$ a Banach space consisting of θ, the solution of (3.8). And give $\mathcal{X}$ a natural Banach structure such that $\{f,\theta^0,\theta^1\} \mapsto \theta$ is an isomorphism from $L^1(0,T;Y_{-1})\times Y_2\times Y_1$ to $\mathcal{X}$. From Proposition 3.4, $\mathcal{X}$ is contained in $C([0,T];Y_2)\cap C^1([0,T];Y_1)$, and (3.9) holds for all θ in $\mathcal{X}$ and χ in $(0,\pi)$.

Assume that $\{f,\theta^0,\theta^1\}$ belongs to $L^1(0,T;Y_{-1})\times Y_2\times Y_1$ and that $\{u,w^0,w^1\}$ belongs to $L^2(0,T)\times Y_2\times Y_1$. Denote by θ the solution of (3.8) and by w the solution of (1.1) given by Theorem 3.1. Denote by $\langle f_1,f_2\rangle$ the linear form between $Y_{-\gamma}$ and Y_γ for any $\gamma\ge 0$ and by $\langle g_1,g_2\rangle_*$ the linear form between $L^\infty(0,T;Y_1)$ and $L^1(0,T;Y_{-1})$. Multiplying (1.1a) by θ and integrating by parts, we obtain

$$\langle w,f\rangle_* + \langle \mathcal{L}w_t(T),\theta^0\rangle + \langle -\mathcal{L}w(T),\theta^1\rangle$$
$$= -\int_0^T u(t)(\theta_x(\eta,t)-\theta_x(\xi,t))\mathrm{d}t + \langle \mathcal{L}w^1,\theta(0)\rangle + \langle -\mathcal{L}w^0,\theta_t(0)\rangle.$$

Now relaxing the assumption of $\{u,w^0,w^1\}$ belonging to $L^2(0,T)\times Y_2\times Y_1$ to $\{u,w^0,w^1\}$ belonging to $(H^1(0,T))'\times Y_1\times Y_0$ and considering the integral $\int_0^T u(t)(\theta_x(\eta,t)-\theta_x(\xi,t))\mathrm{d}t$ as the linear form between u and $\theta_x(\eta,\cdot)-\theta_x(\xi,\cdot)$, we obtain from (3.6) that there exist functions u_0 and u_1 in $L^2(0,T)$ such that

$$\langle w,f\rangle_* + \langle \mathcal{L}w_t(T),\theta^0\rangle + \langle -\mathcal{L}w(T),\theta^1\rangle$$
$$= -\int_0^T [u_0(t)(\theta_x(\eta,t)-\theta_x(\xi,t)) + u_1(t)(\theta_{xt}(\eta,t)-\theta_{xt}(\xi,t))]\mathrm{d}t$$
$$+\langle \mathcal{L}w^1,\theta(0)\rangle + \langle -\mathcal{L}w^0,\theta_t(0)\rangle, \quad (3.14)$$

where $\mathcal{L}w^1$ is in Y_{-2} and $\mathcal{L}w^0$ is in Y_{-1} due to the Lax–Milgram Theorem. Thus $\langle \mathcal{L}w^1,\theta(0)\rangle$ and $\langle -\mathcal{L}w^0,\theta_t(0)\rangle$ are well-defined. Let (3.14) be the definition of the weak solution.

Definition 3.6. Let $T>0$, u in $(H^1(0,T))'$ and (w^0,w^1) in $Y_1\times Y_0$ be given. A weak solution of the problem (1.1) is a function w in $C([0,T];Y_1)$ such that for every $\{f,\theta^0,\theta^1\}$ in $L^1(0,T;Y_{-1})\times Y_2\times Y_1$, (3.14) holds, and $(w(T),w_t(T))$ belongs to $Y_1\times Y_0$.

Step 3. Then we are able to prove the well-posedness of (1.1) when u belongs to $(H^1(0,T))'$.

Theorem 3.7. *Suppose (w^0,w^1) belongs to $Y_1\times Y_0$. For any u in $(H^1(0,T))'$ and for any ξ and η in $(0,\pi)$, the initial and boundary value problem (1.1) admits a unique weak solution w in sense of Definition 3.6. And the map $\{w^0,w^1,u\}\mapsto\{w,w(T),w_t(T)\}$ is continuous and linear with respect to the corresponding norm.*

Proof. Since u belongs to $(H^1(0,T))'$, there exist u_0 and u_1 in $L^2(0,T)$ such that (3.6) holds . Moreover, the map $\{f,\theta^0,\theta^1\}\mapsto\theta$ is an isomorphism from $L^1(0,T;Y_{-1})\times Y_2\times Y_1$ to $\mathcal{X}$. Therefore, we define a linear form Γ on $\mathcal{X}$ such that

$$\Gamma(\theta) = -\int_0^T [u_0(t)(\theta_x(\eta,t)-\theta_x(\xi,t)) + u_1(t)(\theta_{xt}(\eta,t)-\theta_{xt}(\xi,t))]\mathrm{d}t$$
$$+\langle \mathcal{L}w^1,\theta(0)\rangle + \langle -\mathcal{L}w^0,\theta_t(0)\rangle. \quad (3.15)$$

Since (w^0,w^1) belongs to $Y_1\times Y_0$, $(\mathcal{L}w^0,\mathcal{L}w^1)$ belongs to $Y_{-1}\times Y_{-2}$. It follows from Proposition 3.4 that Γ is a continuous linear form on $\mathcal{X}$. Therefore, for the linear form Γ in $\mathcal{X}'$, there exists a unique element $\{w,\zeta_*,\zeta\}$ in $L^\infty(0,T;Y_1)\times Y_{-2}\times Y_{-1}$ such that

$$\langle w,f\rangle_* + \langle \zeta_*,\theta^0\rangle + \langle -\zeta,\theta^1\rangle = \Gamma(\theta) \quad \forall\theta\in\mathcal{X}. \quad (3.16)$$

Next we claim that w above is actually the weak solution of (1.1). It is sufficient to prove that w satisfies (1.1) in weak sense, $\mathcal{L}w(T)=\zeta$ and $\mathcal{L}w_t(T)=\zeta_*$.

Notice that $\{x\mapsto\sin(kx)\}_{k\in\mathbb{N}^*}$ is the family of eigenfunctions of $\mathcal{R}\partial_{xxxx}$. Let $m(x)=\sin(kx)$ for some k in $\mathbb{N}^*$ and h belong to $L^1(0,T)$. Firstly we set $\mathcal{R}f(t)=h(t)m$, $\theta^0=0$ and $\theta^1=0$, and hence $f(t)=h(t)\mathcal{L}m$. Denote by $\lambda^2=k^4/(1+\alpha k^2)$ the corresponding eigenvalue, and take λ positively. Then we obtain from Proposition 3.4 that for t in $(0,T)$, $\theta(t)=q(t)m$, where q satisfies $q_{tt}+\lambda^2 q=h$ and $q(T)=q_t(T)=0$. Clearly we have $q(t)=-\frac{1}{\lambda}\int_t^T\sin(\lambda(t-\sigma))h(\sigma)\mathrm{d}\sigma$. Then we obtain from (3.15) and (3.16) that

$$\int_0^T\langle w,\mathcal{L}m\rangle(q_{tt}+\lambda^2 q)\mathrm{d}t = \langle w,f\rangle_* = \Gamma(\theta)$$
$$= -(m_x(\eta)-m_x(\xi))\int_0^T[u_0(t)q(t)+u_1(t)q_t(t)]\mathrm{d}t$$
$$+q(0)\langle\mathcal{L}w^1,m\rangle - q_t(0)\langle\mathcal{L}w^0,m\rangle. \quad (3.17)$$

Notice that $\langle w,\mathcal{L}m\rangle=\langle\mathcal{L}w,m\rangle$. Then (3.17) implies that

$$\begin{aligned}\langle\mathcal{L}w,m\rangle_{tt}+\lambda^2\langle\mathcal{L}w,m\rangle &= -(m_x(\eta)-m_x(\xi))u,\\ \langle\mathcal{L}w,m\rangle(0) &= \langle\mathcal{L}w^0,m\rangle,\\ \langle\mathcal{L}w,m\rangle_t(0) &= \langle\mathcal{L}w^1,m\rangle.\end{aligned} \quad (3.18)$$

Note that the first equation of (3.18) holds in $(H^1(0,T))'$. Since $m(x)=\sin(kx)$ and $k\ge1$ is an arbitrary natural number, then w satisfies (1.1) in weak sense.

Now we set $f=0$, $\theta^0=0$ and $\theta^1=-m$. Then for t in $(0,T)$, $\theta(t)=\frac{1}{\lambda}\sin(\lambda(T-t))m$. Therefore, (3.15) and (3.16) imply that

$$\langle\zeta,m\rangle=\Gamma(\theta)=\langle\mathcal{L}w^1,m\rangle\frac{\sin(\lambda T)}{\lambda}+\langle\mathcal{L}w^0,m\rangle\cos(\lambda T)$$
$$-(m_x(\eta)-m_x(\xi))\int_0^T\left[u_0(t)\frac{\sin(\lambda(T-t))}{\lambda}-u_1(t)\cos(\lambda(T-t))\right]\mathrm{d}t. \quad (3.19)$$

Moreover, it follows from (3.18) that

$$\langle\mathcal{L}w,m\rangle(T)=\langle\mathcal{L}w^0,m\rangle\cos(\lambda T)+\langle\mathcal{L}w^1,m\rangle\frac{\sin(\lambda T)}{\lambda}$$
$$-\frac{m_x(\eta)-m_x(\xi)}{\lambda}\langle u,\sin(\lambda(T-\cdot))\rangle_{(H^1(0,T))'\times H^1(0,T)}. \quad (3.20)$$

Comparing (3.20) to (3.19), we obtain that $\langle\mathcal{L}w,m\rangle(T)=\langle\zeta,m\rangle$, which proves $\mathcal{L}w(T)=\zeta$.

Next let $f=0$, $\theta^0=m$ and $\theta^1=0$. Then for t in $(0,T)$, $\theta(t)=\cos(\lambda(T-t))m$. Thus (3.15) and (3.16) imply that

$$\langle\zeta_*,m\rangle=\Gamma(\theta)=\langle\mathcal{L}w^1,m\rangle\cos(\lambda T)-\langle\mathcal{L}w^0,m\rangle\lambda\sin(\lambda T)$$
$$-(m_x(\eta)-m_x(\xi))\int_0^T[u_0(t)\cos(\lambda(T-t))+u_1(t)\lambda\sin(\lambda(T-t))]\mathrm{d}t. \quad (3.21)$$

Moreover, (3.18) implies that

$$\langle\mathcal{L}w,m\rangle_t(T)=\langle\mathcal{L}w^1,m\rangle\cos(\lambda T)-\langle\mathcal{L}w^0,m\rangle\lambda\sin(\lambda T)$$
$$-(m_x(\eta)-m_x(\xi))\langle u,\cos(\lambda(T-\cdot))\rangle_{(H^1(0,T))'\times H^1(0,T)}. \quad (3.22)$$

Comparing (3.22) to (3.21), we obtain that $\langle\mathcal{L}w,m\rangle_t(T)=\langle\zeta_*,m\rangle$, which implies $\mathcal{L}w_t(T)=\zeta_*$.

Now we have proved that there exists a unique element $\{w,w(T),w_t(T)\}$ in $L^\infty(0,T;Y_1)\times Y_1\times Y_0$ such that (3.14) holds, and the map $\{w^0,w^1,u\}\mapsto\{w,w(T),w_t(T)\}$ is continuous and linear with respect to the corresponding norm. In fact we have the property of w belonging to $C([0,T];Y_1)$. Since when the known data $\{w^0,w^1,u\}$ belongs to $Y_2\times Y_1\times L^2(0,T)$, we have (3.1). Using a density argument, we conclude the proof of Theorem 3.7. □

4. Proofs of the main results

In this section, we prove the main results. On the one hand, for controllability results, namely Theorem 1.4, we first use the HUM to rewrite the control problem into observability inequality of the adjoint equation. Then we derive the observability inequality by applying the Ingham inequality. The methods for proving Theorem 1.4 are inspired by the ideas and methods used in [6] for Euler–Bernoulli beam with piezoelectric actuator. On the other hand, for non-controllability results, namely Theorems 1.5–1.7, we exhibit initial conditions so that the observability inequalities are false. The approaches in proofs of Theorems 1.5 and 1.6 are inspired by [6]. And the proof of the lack of the controllability in short control time, namely Theorem 1.7, is inspired by the methods used in [31].

4.1. Exact $(H^1)'$-controllability (proof of Theorem 1.4)

We use the HUM to rewrite the controllability problem. Let (ϕ^0,ϕ^1) in $(C^\infty[0,\pi])^2$ satisfy the compatibility conditions (1.2). Denote by ϕ the solution of (3.2) with initial value (ϕ^0,ϕ^1). Let $u_1(t)=\phi_{xt}(\eta,t)-\phi_{xt}(\xi,t)$. Function u_1 belongs to $L^2(0,T)$ due to (3.3). Then define u in $(H^1(0,T))'$ as follows,

$$\langle u,f\rangle_{(H^1(0,T))'\times H^1(0,T)}=\int_0^T u_1 f_t\,\mathrm{d}t,\quad \forall f\in H^1(0,T). \tag{4.1}$$

Then consider the backward system in $(0,\pi)\times(0,T)$

$$\psi_{tt}(x,t)-\alpha\psi_{xxtt}(x,t)+\psi_{xxxx}(x,t)=u(t)\frac{\mathrm{d}}{\mathrm{d}x}[\delta_\eta(x)-\delta_\xi(x)], \tag{4.2a}$$

$$\psi(0,t)=\psi(\pi,t)=\psi_{xx}(0,t)=\psi_{xx}(\pi,t)=0, \tag{4.2b}$$

$$\psi(x,T)=\psi_t(x,T)=0. \tag{4.2c}$$

Problem (4.2) admits a weak solution ψ in $C([0,T];Y_1)$ due to Theorem 3.7. Taking the linear form between (4.2a) and ϕ and integrating by parts, we obtain

$$\begin{aligned}&\langle\mathcal{L}\psi_t(\cdot,0),\phi^0\rangle+\langle-\mathcal{L}\psi(\cdot,0),\phi^1\rangle\\&=\langle u,\phi_x(\eta,\cdot)-\phi_x(\xi,\cdot)\rangle_{(H^1(0,T))'\times H^1(0,T)}\\&=\int_0^T|\phi_{xt}(\eta,t)-\phi_{xt}(\xi,t)|^2\mathrm{d}t.\end{aligned} \tag{4.3}$$

Define the linear operator Λ_* by, for all (ϕ^0,ϕ^1) in $(C^\infty[0,\pi])^2$ satisfy the compatibility conditions (1.2),

$$\Lambda_*(\phi^0,\phi^1)=(\mathcal{L}\psi_t(\cdot,0),-\mathcal{L}\psi(\cdot,0)). \tag{4.4}$$

Since ψ is in $C([0,T];Y_1)$, $(\mathcal{L}\psi_t(\cdot,0),-\mathcal{L}\psi(\cdot,0))$ belongs to $Y_{-2}\times Y_{-1}$, and therefore, the operator Λ_* is well defined. In particular, we have

$$\langle\Lambda_*(\phi^0,\phi^1),(\phi^0,\phi^1)\rangle=\int_0^T|\phi_{xt}(\eta,t)-\phi_{xt}(\xi,t)|^2\mathrm{d}t.$$

Therefore, we can define a seminorm

$$\|(\phi^0,\phi^1)\|_{F_*}:=\left(\int_0^T|\phi_{xt}(\eta,t)-\phi_{xt}(\xi,t)|^2\mathrm{d}t\right)^{\frac12},$$

for all (ϕ^0,ϕ^1) in $(C^\infty[0,\pi])^2$ satisfying the compatibility conditions (1.2).

A classical argument in HUM implies the following proposition.

Proposition 4.1. *All initial data in $Y_{\beta+3}\times Y_{\beta+2}$ are exactly $(H^1)'$-controllable in (ξ,η) at time T if and only if there exists a constant $c>0$ such that*

$$\int_0^T|\phi_{xt}(\eta,t)-\phi_{xt}(\xi,t)|^2\mathrm{d}t\ge c(\|\phi^0\|^2_{H^{-\beta}}+\|\phi^1\|^2_{H^{-\beta-1}}) \tag{4.5}$$

for all (ϕ^0,ϕ^1) in $(C^\infty[0,\pi])^2$ satisfying the compatibility conditions (1.2).

As in the proof of Lemma 3.2, the solution ϕ of the adjoint problem (3.2) has the form of (3.5), which implies

$$\begin{aligned}&\int_0^T|\phi_{xt}(\eta,t)-\phi_{xt}(\xi,t)|^2\mathrm{d}t\\&=4\int_0^T\left|\sum_{k\ge1}k\sin\left(\frac{k(\eta+\xi)}{2}\right)\sin\left(\frac{k(\eta-\xi)}{2}\right)\left\{b_k\cos\left(\frac{k^2t}{\sqrt{1+\alpha k^2}}\right)\right.\right.\\&\left.\left.-\frac{a_kk^2}{\sqrt{1+\alpha k^2}}\sin\left(\frac{k^2t}{\sqrt{1+\alpha k^2}}\right)\right\}\right|^2\mathrm{d}t.\end{aligned} \tag{4.6}$$

To prove the observability inequality (4.5) for some β, we apply the following Ingham inequality (see [11,12]).

Lemma 4.2. *Let $(\nu_k)_{k\in\mathbb{Z}}$ be a strictly increasing sequence of real numbers and γ_∞ be defined by $\gamma_\infty=\liminf_{|k|\to\infty}|\nu_{k+1}-\nu_k|$. Assume that $\gamma_\infty>0$. For any real $T>2\pi/\gamma_\infty$, there exist two constants C_1, $C_2>0$ such that for any sequence $(x_k)_{k\in\mathbb{Z}}$ in $l^2(\mathbb{C})$,*

$$C_1\sum_{k\in\mathbb{Z}}|x_k|^2\le\int_0^T\left|\sum_{k\in\mathbb{Z}}x_ke^{i\nu_kt}\right|^2\mathrm{d}t\le C_2\sum_{k\in\mathbb{Z}}|x_k|^2.$$

We apply Lemma 4.2 with

$$\nu_k=-\nu_{-k}=\frac{k^2}{\sqrt{1+\alpha k^2}},\quad k\in\mathbb{N},$$

$$\begin{aligned}2x_k=2\overline{x_{-k}}&=\left(b_k+i\frac{a_kk^2}{\sqrt{1+\alpha k^2}}\right)\\&\cdot k\sin\left(\frac{k(\eta+\xi)}{2}\right)\sin\left(\frac{k(\eta-\xi)}{2}\right),\quad k\in\mathbb{N}^*,\end{aligned}$$

$$x_0=0.$$

As $\lim_{|k|\to\infty}|\nu_{k+1}-\nu_k|=1/\sqrt{\alpha}$, then for any real $T>2\pi\sqrt{\alpha}$, there exists a constant $C_1>0$ such that

$$\begin{aligned}&\int_0^T|\phi_{xt}(\eta,t)-\phi_{xt}(\xi,t)|^2\mathrm{d}t\\&\ge C_1\sum_{k\ge1}k^2\left(b_k^2+\frac{a_k^2k^4}{1+\alpha k^2}\right)\left[\sin\left(\frac{k(\eta+\xi)}{2}\right)\sin\left(\frac{k(\eta-\xi)}{2}\right)\right]^2.\end{aligned} \tag{4.7}$$

When $\frac{\eta+\xi}{2\pi}$ and $\frac{\eta-\xi}{2\pi}$ belong to A, it follows from (2.1) that there exists a constant $C>0$ such that for all $k\ge1$,

$$\left|\sin\left(\frac{k(\eta\pm\xi)}{2}\right)\right|=\left|\sin\left\{\pi\left[\frac{k(\eta\pm\xi)}{2\pi}-p\right]\right\}\right|\ge\left|\sin\left(\frac{\pi C}{k}\right)\right|\ge\frac{C}{k}. \tag{4.8}$$

Inequalities (4.7) and (4.8) imply that there exists a constant $c>0$ such that

$$\int_0^T|\phi_{xt}(\eta,t)-\phi_{xt}(\xi,t)|^2\mathrm{d}t\ge c\sum_{k\ge1}(a_k^2+b_k^2k^{-2}),$$

which is exactly (4.5) when $\beta=0$. This completes the proof of the first part of Theorem 1.4.

When $\frac{\eta+\xi}{2\pi}$ and $\frac{\eta-\xi}{2\pi}$ belong to B_ε, it follows from (2.2) that there exists a constant $C>0$ such that for all $k\ge1$,

$$\left|\sin\left(\frac{k(\eta\pm\xi)}{2}\right)\right|\ge\frac{C}{k^{1+\varepsilon}}. \tag{4.9}$$

Inequalities (4.7) and (4.9) imply that there exists a constant $c>0$ such that

$$\int_0^T|\phi_{xt}(\eta,t)-\phi_{xt}(\xi,t)|^2\mathrm{d}t\ge c\sum_{k\ge1}(a_k^2k^{-4\varepsilon}+b_k^2k^{-2-4\varepsilon}),$$

which is exactly (4.5) when $\beta=2\varepsilon$. This completes the proof of the second part of Theorem 1.4.

4.2. The condition (1.3) is not sufficient (proof of Theorem 1.5)

In order to prove Theorem 1.5, let us recall the following proposition from the conference paper [22].

Proposition 4.3. *All initial data in $Y_{\beta+3}\times Y_{\beta+2}$ are exactly L^2-controllable in (ξ,η) at time T if and only if there exists a constant $c>0$ such that*

$$\int_0^T|\phi_x(\eta,t)-\phi_x(\xi,t)|^2\mathrm{d}t\ge c(\|\phi^0\|^2_{H^{-\beta}}+\|\phi^1\|^2_{H^{-\beta-1}}) \tag{4.10}$$

for all (ϕ^0,ϕ^1) in $(C^\infty[0,\pi])^2$ satisfying the compatibility conditions (1.2).

We aim to prove the condition (1.3) is not sufficient for any controllability result in this section. From Proposition 4.3, it is sufficient to show that for any $\beta \geq -1$, there exist ξ and η satisfying (1.3) such that (4.10) is false for any $c > 0$. For any $\beta \geq -1$, let

$$\nu > \max\left(\frac{3}{2}\beta + 1, 2\right). \tag{4.11}$$

We choose

$$\frac{\eta+\xi}{2\pi} = \sum_{n=1}^{\infty} \frac{a_n}{10^{n!}}, \tag{4.12}$$

where a_n belongs to $\{0, 1, \ldots, 9\}$ for all $n \geq 1$, and a_n is not identically zero for great n. According to [32] the right-hand side of (4.12) is a Liouville number, i.e., it is transcendental, and there exists a strictly increasing sequence of integers q_n such that

$$\left|\sin\left(q_n \frac{\eta+\xi}{2}\right)\right| \leq \frac{\pi}{q_n^{\nu}} \quad \forall n \geq 1. \tag{4.13}$$

Now we consider the sequence of initial data

$$\phi_n^0(x) = q_n^{\mu}\sin(q_n x), \quad \phi_n^1(x) = 0 \quad \forall x \in (0,\pi), \tag{4.14}$$

where $\mu = \frac{3}{2}\beta$ if $\beta > 0$, and $\mu = 1$ if $-1 \leq \beta \leq 0$. Obviously, (ϕ_n^0, ϕ_n^1) belongs to $(C^\infty[0,\pi])^2$ and satisfies compatibility conditions (1.2) and

$$\|\phi_n^0\|_{H^{-\beta}}^2 + \|\phi_n^1\|_{H^{-\beta-1}}^2 \to \infty. \tag{4.15}$$

Moreover, we obtain from (3.5), (4.11) and (4.13) that

$$\begin{aligned}
&\int_0^T |\phi_{n,x}(\eta,t) - \phi_{n,x}(\xi,t)|^2 \mathrm{d}t \\
&= 4\int_0^T \left| q_n \sin\left(q_n\frac{\eta+\xi}{2}\right)\sin\left(q_n\frac{\eta-\xi}{2}\right) q_n^{\mu}\cos\left(\frac{q_n^2}{\sqrt{1+\alpha q_n^2}}t\right)\right|^2 \mathrm{d}t \\
&\leq 4T q_n^{2(\mu+1)}\left|\sin\left(q_n\frac{\eta+\xi}{2}\right)\right|^2 \\
&\leq 4\pi T q_n^{2(\mu+1-\nu)} \to 0, \quad \text{as } n \to \infty.
\end{aligned} \tag{4.16}$$

Relations (4.15) and (4.16) show that (4.10) is false for any $c > 0$.

Similarly, because of Proposition 4.1, it is sufficient to show that for any $\beta \geq -2$, there exist ξ and η satisfying (1.3) such that (4.5) is false for any $c > 0$. The proof is quite similar to the proof above in this section. For any fixed $\beta \geq -2$, we only need to change ν as

$$\nu > \max\left(\frac{3}{2}\beta + 2, 3\right), \tag{4.17}$$

and to set $\mu = \frac{3}{2}\beta$ if $\beta > 0$, and $\mu = 1$ if $-2 \leq \beta \leq 0$. By similar calculation, we obtain that (4.5) is false for any $c > 0$.

4.3. Non-controllability for initial data in less regular set (proof of Theorem 1.6)

Similarly to the previous section, we aim to prove (4.10) (resp. (4.5)) is false. According to Proposition 2.3, for any ξ and η in $(0,\pi)$, there exists a strictly increasing sequence of positive integers $\{q_n\}_{n\geq 1}$ such that for all $n \geq 1$,

$$\left|\sin\left(q_n\frac{\eta+\xi}{2}\right)\right| \leq \frac{\pi}{\sqrt{q_n}}, \quad \left|\sin\left(q_n\frac{\eta-\xi}{2}\right)\right| \leq \frac{\pi}{\sqrt{q_n}}. \tag{4.18}$$

First we consider the sequence of initial data

$$\phi_n^0(x) = \sin(q_n x), \quad \phi_n^1(x) = 0 \quad \forall x \in (0,\pi). \tag{4.19}$$

We note that for any $\varepsilon > 0$,

$$\|\phi_n^0\|_{H^\varepsilon}^2 + \|\phi_n^1\|_{H^{\varepsilon-1}}^2 = Cq_n^{2\varepsilon} \to \infty, \quad n \to \infty, \tag{4.20}$$

where C is a positive constant. By (3.5) and (4.18) we have that for all $n \geq 1$,

$$\begin{aligned}
&\int_0^T |\phi_{n,x}(\eta,t) - \phi_{n,x}(\xi,t)|^2 \mathrm{d}t \\
&= 4\int_0^T \left| q_n \sin\left(q_n\frac{\eta+\xi}{2}\right)\sin\left(q_n\frac{\eta-\xi}{2}\right) \right. \\
&\quad \left. \cdot \cos\left(\frac{q_n^2}{\sqrt{1+\alpha q_n^2}}t\right)\right|^2 \mathrm{d}t \leq K,
\end{aligned} \tag{4.21}$$

where K is a positive constant. So (4.20) and (4.21) show that (4.10) is false for $\beta = -\varepsilon$ and arbitrary $c > 0$.

Then we choose the sequence of initial data

$$\phi_n^0(x) = q_n^{-1}\sin(q_n x), \quad \phi_n^1(x) = 0 \quad \forall x \in (0,\pi). \tag{4.22}$$

We note that for any $\varepsilon > 0$,

$$\|\phi_n^0\|_{H^{\varepsilon+1}}^2 + \|\phi_n^1\|_{H^{\varepsilon}}^2 = Cq_n^{2\varepsilon} \to \infty, \quad n \to \infty, \tag{4.23}$$

where C is a positive constant. By (3.5) and (4.18) we have that for all $n \geq 1$,

$$\begin{aligned}
&\int_0^T |\phi_{n,xt}(\eta,t) - \phi_{n,xt}(\xi,t)|^2 \mathrm{d}t \\
&= 4\int_0^T \left| \sin\left(q_n\frac{\eta+\xi}{2}\right)\sin\left(q_n\frac{\eta-\xi}{2}\right) \right. \\
&\quad \left. \cdot \frac{q_n^2}{\sqrt{1+\alpha q_n^2}} \sin\left(\frac{q_n^2}{\sqrt{1+\alpha q_n^2}}t\right)\right|^2 \mathrm{d}t \leq K,
\end{aligned} \tag{4.24}$$

where K is a positive constant. So (4.23) and (4.24) show that (4.5) is false for $\beta = -\varepsilon - 1$ and arbitrary $c > 0$.

4.4. The lack of controllability in short control time (proof of Theorem 1.7)

We prove the lack of $(H^1)'$-controllability when $0 < T < 2\pi\sqrt{\alpha}$ in this section. As for the proof of the lack of L^2-controllability, one can find in [22]. Let $0 < T < 2\pi\sqrt{\alpha}$ and ξ, η in $(0,\pi)$ be arbitrary. For any $\beta \geq -2$, we aim to find $\{(\phi_m^0, \phi_m^1)\}_{m\in\mathbb{N}^*}$ such that

$$\int_0^T |\phi_{m,xt}(\eta,t) - \phi_{m,xt}(\xi,t)|^2 \mathrm{d}t \to 0, \quad \text{as } m \to \infty,$$

and

$$\|\phi_m^0\|_{H^{-\beta}}^2 + \|\phi_m^1\|_{H^{-\beta-1}}^2 \geq c > 0$$

for any $m \geq 1$. As in Section 4.1, we denote

$$\lambda_n = -\lambda_{-n} = \frac{n^2}{\sqrt{1+\alpha n^2}}, \quad n \in \mathbb{N}^*. \tag{4.25}$$

Obviously, $\{\lambda_n\}_{n\in\mathbb{Z}^*}$ is a strictly increasing sequence, and $\lim_{|n|\to\infty}|\lambda_{n+1} - \lambda_n| = 1/\sqrt{\alpha} > 0$. Adding or subtracting finite numbers in the sequence does not affect the result of Proposition 2.4, so we can apply Proposition 2.4 to the sequence $\{\lambda_n\}_{n\in\mathbb{Z}^*}$. Define $N(x,r)$ as in Proposition 2.4 corresponding to $\{\lambda_n\}_{n\in\mathbb{Z}^*}$. We have the following lemma.

Lemma 4.4. *Let $\{\lambda_n\}_{n\in\mathbb{Z}^*}$ and $N(x,r)$ be defined above. We have that*

$$\frac{N(x,r)}{r} \to \sqrt{\alpha}, \quad \text{as } r \to \infty \tag{4.26}$$

holds uniformly relative to x in $\mathbb{R}$.

We will prove this lemma in Appendix. As $0 < T < 2\pi\sqrt{\alpha}$, we can choose T' such that $0 < T < T' < 2\pi\sqrt{\alpha}$. Let f in $L^2(0, 2\pi\sqrt{\alpha})$ be a real valued function such that $f(t) = 0$ if $0 \leq t \leq T$ and $\|f\|_{L^2(0,T')} \neq 0$. According to Lemma 4.4 and Proposition 2.4, the family $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}^*}$ contains a subfamily $\{e^{i\lambda_{q_n} t}\}_{n\in\mathbb{Z}^*}$ which forms a Riesz basis in $L^2(0,T')$. Moreover, the subsequence $\{\lambda_{q_n}\}_{n\in\mathbb{Z}}$ satisfies $\lambda_{q_n} = -\lambda_{q_{-n}}$. Then for the function f in $L^2(0,T')$ defined above, there exists a sequence $\{l_n\}_{n\in\mathbb{Z}^*}$ in $l^2(\mathbb{C})$ such that $f(t) = \sum_{n\in\mathbb{Z}^*} l_n e^{i\lambda_{q_n} t}$ holds in $L^2(0,T')$, and $0 < \sum_{n\in\mathbb{Z}^*}|l_n|^2 < \infty$. Since $f(t)$ is a real valued function, we have $l_n = \overline{l_{-n}}$.

Now we can define the sequence $\{(\phi^0_m, \phi^1_m)\}_{m\in\mathbb{N}^*}$ of initial data as the following,

$$\phi^0_m(x) = 2\sum_{n=1}^{m} \mathrm{Re}(l_n)\left[q_n \sin\left(q_n\frac{\eta+\xi}{2}\right)\right.$$
$$\left.\cdot \sin\left(q_n\frac{\eta-\xi}{2}\right)\right]^{-1} \frac{\sqrt{1+\alpha q_n^2}}{q_n^2}\sin(q_n x),$$
$$\phi^1_m(x) = -2\sum_{n=1}^{m} \mathrm{Im}(l_n)\left[q_n \sin\left(q_n\frac{\eta+\xi}{2}\right)\right.$$
$$\left.\cdot \sin\left(q_n\frac{\eta-\xi}{2}\right)\right]^{-1} \sin(q_n x). \tag{4.27}$$

Notice that (1.3) holds. Consequently, the sequence $\{(\phi^0_m, \phi^1_m)\}_{m\in\mathbb{N}^*}$ of initial data is well-defined.

Since $0 < \sum_{n\in\mathbb{Z}^*}|l_n|^2 < \infty$ and $l_n = \overline{l_{-n}}$, there exists $m_0 \geq 1$ such that $l_{m_0} \neq 0$. So for $m \geq m_0$ and for any $\beta \geq -2$,

$$\|\phi^0_m\|^2_{H^{-\beta}} + \|\phi^1_m\|^2_{H^{-\beta-1}} \geq \|\phi^0_{m_0}\|^2_{H^{-\beta}} + \|\phi^1_{m_0}\|^2_{H^{-\beta-1}} = c > 0. \tag{4.28}$$

Moreover, thanks to (3.5), we have

$$\int_0^T |\phi_{m,xt}(\eta,t) - \phi_{m,xt}(\xi,t)|^2 \mathrm{d}t = 4\int_0^T \left|\sum_{n=-m,n\neq 0}^{m} l_n e^{\lambda_{q_n} t}\right|^2 \mathrm{d}t. \tag{4.29}$$

Since $0 = f(t) = \sum_{n\in\mathbb{Z}^*} l_n e^{\lambda_{q_n} t}$ in $L^2(0,T)$, we obtain that

$$\int_0^T |\phi_{m,xt}(\eta,t) - \phi_{m,xt}(\xi,t)|^2 \mathrm{d}t \to 0, \quad \text{as } m\to\infty. \tag{4.30}$$

Relations (4.28) and (4.30) finish the proof of the lack of $(H^1)'$-controllability for any $\beta \geq -2$.

5. Conclusion

The exact controllability problem for Rayleigh beam equation with piezoelectric actuator has been fully considered. The exact controllability in less regular spaces, namely $(H^1)'$-controllability, are investigated. Moreover, several non-controllability results are proved. Especially, minimal control time for the exact controllability is deduced from the non-controllability result in short control time. As written in Remark 1.8, exact controllability in critical time is an open problem. Controllability problem for other types of beam equation with piezoelectric actuator, such as shear beam equation, remains also open.

CRediT authorship contribution statement

Yubo Bai: Methodology, Writing – original draft. **Christophe Prieur:** Supervision, Writing – review & editing. **Zhiqiang Wang:** Supervision, Writing – review & editing.

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

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Appendix. Proof of Lemma 4.4

Solving $\frac{k^2}{\sqrt{1+\alpha k^2}} < r$, we obtain that $k < [(\alpha r^2 + r\sqrt{\alpha^2 r^2+4})/2]^{\frac{1}{2}}$. Thus we define a real function $g : (0,+\infty) \to (0,+\infty)$ as $g(r) = [(\alpha r^2 + r\sqrt{\alpha^2 r^2+4})/2]^{\frac{1}{2}}$ and $\{x\} = q(x) - 1$, where $q(x) = \min_{q\in\mathbb{Z}}\{q \geq x\}$. Obviously, $x - 1 \leq \{x\} < x$. Then we have $N(0,r) = \{g(r)\}$. Notice that for $x \geq 0$, $N(x,r) = N(0,x+r) - N(0,x) = \{g(x+r)\} - \{g(x)\}$. Therefore, we have $g(x+r) - g(x) - 1 \leq N(x,r) \leq g(x+r) - g(x) + 1$. Now we need to estimate $g(x+r) - g(x)$ for $x, r > 0$.

Let $f(x) = \frac{x^2}{\sqrt{1+\alpha x^2}}$ for $x > 0$. Then simple calculation shows that $0 < x < \sqrt{2/\alpha}$ implies $f''(x) > 0$. Therefore, there exists N_0 in $\mathbb{N}$ satisfying $N_0 > \sqrt{2/\alpha} + 1$ such that for all $n \geq N_0$, $\lambda_{n+1} - \lambda_n$ decreases and converges to $1/\sqrt{\alpha}$.

Then for $x \geq N_0$, $N(N_0, r) \leq N(x,r) \leq \lim_{x\to+\infty} N(x,r)$. Simple calculation shows that $\lim_{x\to+\infty}[g(x+r) - g(x)] = \sqrt{\alpha}r$. Therefore, we obtain that $g(N_0+r) - g(N_0) - 1 \leq N(x,r) \leq \sqrt{\alpha}r + 1$ holds for $x \geq N_0$.

For $0 \leq x \leq N_0$, $N(x,r) \leq N(0,N_0) + N(N_0,r) \leq \sqrt{\alpha}r + 1 + g(N_0)$. Assuming that $r > N_0$, we have

$$N(x,r) \geq N(N_0, r - N_0 + x) = N(0, x+r) - N(0,N_0)$$
$$\geq g(x+r) - g(N_0) - 1 \geq \min_{x\in[0,N_0]} g(x+r) - g(N_0) - 1.$$

Then for all $x \geq 0$ and $r > N_0$, $\min_{x\in[0,N_0]} g(x+r) - g(N_0) - 1 \leq N(x,r) \leq \sqrt{\alpha}r + 1 + g(N_0)$, and hence

$$\lim_{r\to\infty}\frac{N(x,r)}{r} = \sqrt{\alpha} \tag{A.1}$$

holds uniformly relative to $x \geq 0$.

For $-r \leq x < 0$,

$$N(0,|x|) + N(0, r-|x|) \leq N(x,r) \leq N(0,|x|) + N(0,r-|x|) + 1.$$

Let θ belong to $[0,1]$ and $|x| = \theta r$. Note that $N(0,|x|) + N(0, r - |x|) = \{g(\theta r)\} + \{g((1-\theta)r)\}$. We obtain from the expression of g that $\lim_{r\to\infty} g(\theta r)/r = \sqrt{\alpha}\theta$. Consequently, (A.1) holds uniformly relative to $-r \leq x < 0$.

For $x < -r$, $N(|x|-r, r) \leq N(x,r) \leq N(|x|-r, r) + 1$. Let $t = |x| - r \geq 0$. Then as same as in the situation $x \geq 0$, we have that (A.1) holds uniformly relative to $t \geq 0$, which means that (A.1) holds uniformly relative to $x < -r$.

Combining all the situations, we have that (A.1) holds uniformly relative to x in $\mathbb{R}$. Lemma 4.4 is thus proved.

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