A Note on Controllability and non-Controllability for a Rayleigh Beam with Piezoelectric Actuator*

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Abstract

In this paper, the exact controllability problem for a Rayleigh beam with piezoelectric actuator is considered. The main contributions of this work are to give the exact controllability results and to give the minimal controllability time. Controllability results show that the space of controllable initial data depends on the location of the actuator. The approach to prove controllability results is based on Hilbert Uniqueness Method and some results on the theory of Diophantine approximation. Due to the rotary inertia term in the Rayleigh beam equation, Rayleigh beam equation possesses finite propagation speed, and consequently the controllability results hold when the control time surpasses a critical time. This critical time is proved to be the minimal controllability time by using the Riesz basis property of exponential family in $L^2(0,T)$. The controllability in critical time is still an open problem.

1 Problem statement and main results

We consider the control problem modelling the transverse deflection of a Rayleigh beam which is subject to the action of an attached piezoelectric actuator. Assuming that the beam is hinged at both ends, the equation of Rayleigh beam can be written as (see, for instance, [7, 8]), for (x,t) in $(0,\pi)\times(0,+\infty)$,

$$w_{tt}(x,t) - \alpha w_{xxtt}(x,t) + w_{xxxx}(x,t) = u(t)\frac{\mathrm{d}}{\mathrm{d}x}[\delta_\eta(x) - \delta_\xi(x)], \tag{1.1a}$$

$$w(0,t) = w(\pi,t) = w_{xx}(0,t) = w_{xx}(\pi,t) = 0, \tag{1.1b}$$

$$w(x,0) = w^0(x),\ w_t(x,0) = w^1(x). \tag{1.1c}$$

In the equations above w represents the transverse deflection of the beam, $\alpha > 0$ is a physical constant, ξ and η stand for the ends of the actuator ($0<\xi<\eta<\pi$), and δ_y is the Dirac mass at the point y. The control is given by the function $u:[0,T]\to\mathbb{R}$ standing for the time variation of the voltage applied to the actuator.

Our main purpose is to find the initial data that can be steered to rest by means of the control function u. To give the precise definitions of exact controllability, let us introduce for any ω in $\mathbb{R}$ the functional space Y_ω as follows. Let $Y_0 = L^2(0,\pi)$. For $\omega>0$, let Y_ω be the closure in $H^\omega(0,\pi)$ of the set of y in $C^\infty([0,\pi])$ satisfying the conditions

$$y^{(2n)}(0) = y^{(2n)}(\pi) = 0 \quad \forall n \geq 0. \tag{1.2}$$

For $\omega<0$, let Y_ω be the dual space of $Y_{-\omega}$ with respect to the space Y_0. Then we give the precise definitions.

Definition 1.1. The initial data (w^0,w^1) in $Y_2\times Y_1$ is exactly L^2-controllable in (ξ,η) at time T if there exists u in $L^2(0,T)$ such that the solution w of (1.1) satisfies the condition $w(\cdot,T) = w_t(\cdot,T) = 0$.

Note that in Definition 1.1, the space where the initial data (w^0,w^1) can be taken depends on the well-posedness of (1.1) (see Section 3). Notice that the system (1.1) is a time-reversible linear system, so the exact controllability is equivalent to the null controllability (see Theorem 2.41 of [6, p. 55]).

In order to state the exact controllability results, let $\varepsilon>0$ and let the sets $A\subset(0,1)$ and $B_\varepsilon\subset(0,1)$ be the sets defined in Section 2. From Section 2, the set A is uncountable and has zero Lebesgue measure and the Lebesgue measure of set B_ε is 1.

Our exact controllability results are the following.

Theorem 1.1. *Let $T>2\pi\sqrt{\alpha}$ and $\varepsilon>0$.*

1. *Suppose that $\frac{\eta+\xi}{2\pi}$ and $\frac{\eta-\xi}{2\pi}$ belong to the set A. Then all initial data in $Y_4\times Y_3$ are exactly L^2-controllable in (ξ,η) at time T.*
2. *Suppose that $\frac{\eta+\xi}{2\pi}$ and $\frac{\eta-\xi}{2\pi}$ belong to the set B_ε. Then all initial data in $Y_{4+2\varepsilon}\times Y_{3+2\varepsilon}$ are exactly L^2-controllable in (ξ,η) at time T.*

Theorem 1.1 gives us two exact L^2-controllability results. The first result of Theorem 1.1 shows that, for

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the end of the piezoelectric actuator in an uncountable zero measure set, we have the exact L^2-controllability in space $Y_4 \times Y_3$. The second result of Theorem 1.1 shows that, for almost all choices of the end of the piezoelectric actuator, we have the exact L^2-controllability in more regular Sobolev spaces than $Y_4 \times Y_3$.

Notice that the exact controllability results in Theorem 1.1 require $T > 2\pi\sqrt{\alpha}$, however, in [15], the exact controllability results for Euler-Bernoulli beam have no requirement of control time. Consequently, a huge difference between Rayleigh beam and Euler-Bernoulli beam is revealed and the reason lies in various distributions of their eigenvalues. More precisely, under the same boundary condition (1.1b), the eigenvalues of Rayleigh beam equation are $\frac{k^4}{1+\alpha k^2}$ for k in $\mathbb{N}^*$ (see Section 3) while the eigenvalues of Euler-Bernoulli beam equation are k^4 for k in $\mathbb{N}^*$ (see [15]). Roughly speaking, this fact implies that Rayleigh beam equation possesses finite propagation speed and that Euler-Bernoulli beam equation possesses infinite propagation speed. For this reason, the exact controllability results for Rayleigh beam require $T > 2\pi\sqrt{\alpha}$ while the exact controllability results for Euler-Bernoulli beam hold for all $T > 0$ (see [15]). Based on this fact, we give the non-controllability result for $0 < T < 2\pi\sqrt{\alpha}$.

THEOREM 1.2. *Let $0 < T < 2\pi\sqrt{\alpha}$ and ξ, η in $(0, \pi)$ be arbitrary. For any $\varepsilon \geq 0$, the space $Y_{2+\varepsilon} \times Y_{1+\varepsilon}$ contains initial data that are not exactly L^2-controllable in (ξ, η) at time T.*

From Theorem 1.2, we can see that $T \geq 2\pi\sqrt{\alpha}$ is necessary for exact controllability for Rayleigh beam equation. Therefore, minimal control time for the exact controllability is obtained. As far as we know, this is the first result stating a lack of controllability for Rayleigh beam in short control time.

These two theorems characterize the exact L^2-controllability for (1.1). Minimal control time is revealed through these two results. The exact controllability with less regular control function and the other non-controllability results are exhibited in the full version of this paper [3]. The exact controllability in critical time, i.e. $T = 2\pi\sqrt{\alpha}$, is still an open problem.

2 Preliminaries

In this section, we provide some known results on the theory of Diophantine approximation (see [5, 10]) and the Riesz basis property of exponential family (see [1]).

For a real number ρ, we denote by $\|\rho\|_{\mathbb{Z}}$ the difference, taken positively, between ρ and the nearest integer, i.e.,

$$\|\rho\|_{\mathbb{Z}} = \min_{n\in\mathbb{Z}} |\rho - n|.$$

Let us denote by A the set of all irrationals ρ in $(0, 1)$ such that if $[0, a_1, \ldots, a_n \ldots]$ is the expansion of ρ as a continued fraction, then (a_n) is bounded. The set A is uncountable and its Lebesgue measure is equal to zero (see Theorem I of [5, p. 120]). The following property proven in Theorem 6 of [10, p. 23] is essential for this paper.

PROPOSITION 2.1. *A number ρ is in A if and only if there exists a constant $C > 0$ such that*

$$\|q\rho\|_{\mathbb{Z}} \geq \frac{C}{q} \tag{2.1}$$

for all strictly positive integer q.

The next proposition, which is proved in [5, p. 120], shows that an inequality slightly weaker than (2.1) holds for almost all points in $(0, 1)$.

PROPOSITION 2.2. *For any $\varepsilon > 0$ there exists a set $B_\varepsilon \subset (0, 1)$ having Lebesgue measure equal to 1 and a constant $C > 0$, such that for any ρ in B_ε,*

$$\|q\rho\|_{\mathbb{Z}} \geq \frac{C}{q^{1+\varepsilon}} \tag{2.2}$$

for all strictly positive integer q.

The next proposition proven in Theorem II.4.18 of [1, p. 109] on the Riesz basis property of exponential family in $L^2(0, T)$ is essential to prove Theorem 1.2.

PROPOSITION 2.3. *Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of complex numbers such that*

$$\sup_{n\in\mathbb{Z}} |\mathrm{Im}\lambda_n| < \infty, \quad \inf_{n\neq m} |\lambda_m - \lambda_n| > 0.$$

Let

$$N(x, r) := \sharp\{\lambda_n | x \leq \mathrm{Re}\lambda_n < x + r\}, \quad x \in \mathbb{R}, r > 0,$$

where $\sharp\mathcal{A}$ is the number of elements in the set $\mathcal{A}$. Assume that for some $T > 0$,

$$\lim_{r\to\infty} \frac{N(x, r)}{r} = \frac{T}{2\pi}$$

holds uniformly relative to $x \in \mathbb{R}$. Then for any T' in $(0, T)$, $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ contains a subfamily $\{e^{i\lambda_{q_n} t}\}_{n\in\mathbb{Z}}$ that forms a Riesz basis in $L^2(0, T')$. Moreover, if $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a sequence of real numbers such that $\lambda_n = -\lambda_{-n}$, the subsequence $\{\lambda_{q_n}\}_{n\in\mathbb{Z}}$ satisfies $\lambda_{q_n} = -\lambda_{q_{-n}}$.

3 Well-posedness of (1.1)

The well-posedness result of (1.1) has been proved in [16]. We state it and show the proof here, because the process of the proof is also used in the next section.

Theorem 3.1. *Suppose that* (w^0, w^1) *belongs to* $Y_2 \times Y_1$. *For any* u *in* $L^2(0,T)$ *and for any* ξ *and* η *in* $(0,\pi)$, *the initial and boundary value problem* (1.1) *admits a unique solution* w *having the regularity*

$$w \in C([0,T]; Y_2) \cap C^1([0,T]; Y_1). \tag{3.1}$$

In order to prove Theorem 3.1, let us first consider the adjoint problem of (1.1) in $(0,\pi) \times (0,+\infty)$,

$$\phi_{tt}(x,t) - \alpha\phi_{xxtt}(x,t) + \phi_{xxxx}(x,t) = 0, \tag{3.2a}$$
$$\phi(0,t) = \phi(\pi,t) = \phi_{xx}(0,t) = \phi_{xx}(\pi,t) = 0, \tag{3.2b}$$
$$\phi(x,0) = \phi^0(x),\ \phi_t(x,0) = \phi^1(x). \tag{3.2c}$$

The following lemma proved in [16] shows the well-posedness of the adjoint problem (3.2) and one trace regularity needed in the proof of Theorems 1.1 and 3.1.

Lemma 3.1. *For any initial data* (ϕ^0, ϕ^1) *in* $Y_2 \times Y_1$, *there exists a unique weak solution* ϕ *of* (3.2) *in the class* $C([0,T]; Y_2) \cap C^1([0,T]; Y_1)$. *Moreover, for all* χ *in* $(0,\pi)$, $\phi_x(\chi,\cdot)$ *belongs to* $L^2(0,T)$ *and there exists* $C > 0$ *such that*

$$\|\phi_x(\chi,\cdot)\|^2_{L^2(0,T)} \le C(\|\phi^0\|^2_{H^1(0,\pi)} + \|\phi^1\|^2_{L^2(0,\pi)}). \tag{3.3}$$

Proof. It is easy to see, by the semigroup method, that the problem (3.2) is well-posed in the space $Y_2 \times Y_1$ (see [14, p. 104]).

Next we prove (3.3). Since the family of functions $\{x \mapsto \sin(kx)\}_{k\in\mathbb{N}^*}$ is the orthogonal basis of Y_1 and Y_2 respectively, let

$$\phi^0(x) = \sum_{k\ge1} a_k \sin(kx), \quad \phi^1(x) = \sum_{k\ge1} b_k \sin(kx)$$

with $(k^2 a_k)$ and (kb_k) in $l^2(\mathbb{R})$. By standard computation, we have

$$\phi(x,t) = \sum_{k\ge1}\left[a_k \cos\left(\frac{k^2}{\sqrt{1+\alpha k^2}}t\right) + \frac{b_k\sqrt{1+\alpha k^2}}{k^2}\sin\left(\frac{k^2}{\sqrt{1+\alpha k^2}}t\right)\right]\sin(kx). \tag{3.4}$$

Then for all $T > 0$, $\phi_x(\chi,\cdot)$ belongs to $L^2(0,T)$ and

$$\int_0^T |\phi_x(\chi,t)|^2 \mathrm{d}x \le C\sum_{k\ge1}(a_k^2 k^2 + b_k^2),$$

which yields (3.3). ∎

Proof. [Proof of Theorem 3.1] Thanks to Lemma 3.1, the following backward adjoint problem in $(0,\pi)\times(0,\tau)$ is well-posed in $Y_2 \times Y_1$ for every $\tau > 0$ and g in Y_1.

$$v_{tt}(x,t) - \alpha v_{xxtt}(x,t) + v_{xxxx}(x,t) = 0, \tag{3.5a}$$
$$v(0,t) = v(\pi,t) = v_{xx}(0,t) = v_{xx}(\pi,t) = 0, \tag{3.5b}$$
$$v(x,\tau) = 0,\ v_t(x,\tau) = g(x). \tag{3.5c}$$

Moreover, for any χ in $(0,\pi)$,

$$\|v_x(\chi,\cdot)\|_{L^2(0,\tau)} \le C\|g\|_{Y_0}. \tag{3.6}$$

Since (1.1a) is linear and Lemma 3.1 holds, it is enough to consider the case $w^0 = w^1 = 0$. Suppose that g belongs to $C_0^\infty(0,\pi)$, and let v be the solution of (3.5). Define a linear operator $\mathcal{L} := I - \alpha\partial_{xx}$. It is well-known that the operator $\mathcal{L}$ is an isomorphism from Y_2 to Y_0 and an isomorphism from Y_1 to Y_{-1} by Lax-Milgram Theorem. If we multiply (1.1a) by v and integrate by parts, we obtain

$$\int_0^\pi \mathcal{L}w(x,\tau)g(x)\mathrm{d}x = \int_0^\tau u(t)(v_x(\eta,t) - v_x(\xi,t))\mathrm{d}t. \tag{3.7}$$

Inequality (3.6) implies that

$$\left|\int_0^\tau u(t)(v_x(\eta,t) - v_x(\xi,t))\mathrm{d}t\right| \le 2C\|u\|_{L^2(0,T)}\|g\|_{Y_0},$$

so by (3.7), we obtain $\mathcal{L}w(\cdot,\tau)$ belongs to Y_0, and hence $w(\cdot,\tau)$ belongs to Y_2, for all τ in $[0,T]$. By replacing τ by $\tau+h$ in (3.7) we easily get that

$$w \in C([0,T]; Y_2). \tag{3.8}$$

Denote $\mathcal{R} := (I - \alpha\partial_{xx})^{-1}$. It follows from Lax-Milgram Theorem that the operator $\mathcal{R}$ is an isomorphism from Y_{-2} to Y_0 and an isomorphism from Y_{-1} to Y_1. Applying $\mathcal{R}$ to both sides of (1.1a) yields

$$w_{tt}(x,t) + \mathcal{R}w_{xxxx}(x,t) = u(t)\mathcal{R}\frac{\mathrm{d}}{\mathrm{d}x}[\delta_\eta(x) - \delta_\xi(x)]. \tag{3.9}$$

Regularity (3.8) implies that

$$\mathcal{R}w_{xxxx} \in C([0,T]; Y_0). \tag{3.10}$$

As w satisfies (3.9) and $\frac{\mathrm{d}\delta_b}{\mathrm{d}x}$ belongs to Y_{-2} for all b in $(0,\pi)$, we obtain from (3.10) that

$$w_{tt} \in L^2(0,T; Y_0). \tag{3.11}$$

Applying the intermediate derivative theorem (see Theorem 2.3 of [13, p. 15]) with (3.8) and (3.11), it follows that

$$w_t \in L^2(0,T; Y_1). \tag{3.12}$$

The conclusion (3.1) is now a consequence of (3.8) and (3.12) and of the general lifting result from [11]. ∎

4 Proofs of the main results

In this section, we prove the main results. On the one hand, for controllability results, namely Theorem 1.1, we first use the HUM, introduced in [12], to rewrite the control problem into observability inequality of the adjoint equation. Then we derive the observability inequality by applying the Ingham inequality. The methods for proving Theorem 1.1 are inspired by the ideas and methods used in [15] for Euler-Bernoulli beam with piezoelectric actuator. On the other hand, for non-controllability in short time, namely Theorem 1.2, we exhibit initial conditions so that the observability inequality is false. The approach for proving Theorem 1.2 is inspired by the methods used in [2].

4.1 Exact L^2-controllability (Proof of Theorem 1.1) Let (ϕ^0, ϕ^1) in $(C^\infty[0,\pi])^2$ satisfying the compatibility conditions (1.2) and denote by $\phi(x,t)$ the solution of (3.2) with initial value (ϕ^0, ϕ^1).

Consider a backward adjoint system in $(0,\pi)\times(0,T)$

$$\begin{aligned}
&\text{(4.1a)} \quad && \psi_{tt}(x,t) - \alpha\psi_{xxtt}(x,t) + \psi_{xxxx}(x,t) \\
& && \qquad = u(t)\frac{\mathrm{d}}{\mathrm{d}x}[\delta_\eta(x) - \delta_\xi(x)], \\
&\text{(4.1b)} \quad && \psi(0,t) = \psi(\pi,t) = \psi_{xx}(0,t) = \psi_{xx}(\pi,t) = 0, \\
&\text{(4.1c)} \quad && \psi(x,T) = \psi_t(x,T) = 0,
\end{aligned}$$

where u in $L^2(0,T)$ will be chosen later. Problem (4.1) is well-posed according to Theorem 3.1. Then, multiplying (4.1a) by ϕ and integrating by parts, we get

$$\text{(4.2)} \quad \int_0^\pi \phi^0(x)\mathcal{L}\psi_t(x,0) - \phi^1(x)\mathcal{L}\psi(x,0)\mathrm{d}x = \int_0^T u(t)(\phi_x(\eta,t) - \phi_x(\xi,t))\mathrm{d}t.$$

Let $u(t) = \phi_x(\eta,t) - \phi_x(\xi,t)$. Since (3.3), u belongs to $L^2(0,T)$. Define a linear operator Λ satisfying

$$\Lambda(\phi^0,\phi^1) = (\mathcal{L}\psi_t(\cdot,0), -\mathcal{L}\psi(\cdot,0)).$$

Since $(\mathcal{L}\psi_t(\cdot,0), -\mathcal{L}\psi(\cdot,0))$ belongs to $Y_{-1}\times Y_0$ by Theorem 3.1, the operator Λ is well defined. In particular, we obtain from (4.2) that

$$\langle \Lambda(\phi^0,\phi^1), (\phi^0,\phi^1)\rangle = \int_0^T |\phi_x(\eta,t) - \phi_x(\xi,t)|^2\mathrm{d}t.$$

Therefore, we can define a seminorm

$$\|(\phi^0,\phi^1)\|_F := \left(\int_0^T |\phi_x(\eta,t) - \phi_x(\xi,t)|^2\mathrm{d}t\right)^{\frac{1}{2}},$$

for all (ϕ^0,ϕ^1) in $(C^\infty[0,\pi])^2$ satisfying the compatibility conditions (1.2).

A classical argument in HUM implies the following proposition.

Proposition 4.1. *All initial data in $Y_{\beta+3}\times Y_{\beta+2}$ are exactly L^2-controllable in (ξ,η) at time T if and only if there exists a constant $c>0$ such that*

$$\text{(4.3)} \quad \int_0^T |\phi_x(\eta,t) - \phi_x(\xi,t)|^2\mathrm{d}t \geq c(\|\phi^0\|^2_{H^{-\beta}} + \|\phi^1\|^2_{H^{-\beta-1}})$$

for all (ϕ^0,ϕ^1) in $(C^\infty[0,\pi])^2$ satisfying the compatibility conditions (1.2).

Equation (4.3) is called observability inequality. As in the proof of Lemma 3.1, the solution ϕ of the adjoint problem (3.2) has the form of (3.4), which implies that

$$\begin{aligned}
\text{(4.4)} \quad & \int_0^T |\phi_x(\eta,t) - \phi_x(\xi,t)|^2\mathrm{d}t \\
&= 4\int_0^T \Bigg| \sum_{k\geq 1} k \sin\left(\frac{k(\eta+\xi)}{2}\right)\sin\left(\frac{k(\eta-\xi)}{2}\right) \\
&\quad \cdot \left\{ a_k \cos\left(\frac{k^2}{\sqrt{1+\alpha k^2}}t\right) \right. \\
&\quad \left. + \frac{b_k\sqrt{1+\alpha k^2}}{k^2}\sin\left(\frac{k^2}{\sqrt{1+\alpha k^2}}t\right)\right\}\Bigg|^2 \mathrm{d}t.
\end{aligned}$$

To prove observability inequality (4.3) for some β, we apply the following Ingham inequality (see [4, 9]) to our problem.

Lemma 4.1. *Let $(\nu_k)_{k\in\mathbb{Z}}$ be a strictly increasing sequence of real numbers and let γ_∞ be defined by*

$$\gamma_\infty = \liminf_{|k|\to\infty} |\nu_{k+1} - \nu_k|.$$

Assume that $\gamma_\infty > 0$. For any real $T > 2\pi/\gamma_\infty$, there exist two constants C_1, $C_2 > 0$ such that for any sequence $(x_k)_{k\in\mathbb{Z}}$ in $l^2(\mathbb{C})$,

$$C_1\sum_{k\in\mathbb{Z}}|x_k|^2 \leq \int_0^T \left|\sum_{k\in\mathbb{Z}} x_k \mathrm{e}^{i\nu_k t}\right|^2 \mathrm{d}t \leq C_2\sum_{k\in\mathbb{Z}}|x_k|^2.$$

We apply Lemma 4.1 with

$$\begin{aligned}
&\nu_k = -\nu_{-k} = \frac{k^2}{\sqrt{1+\alpha k^2}}, \quad k\in\mathbb{N}, \\
&2x_k = 2\overline{x_{-k}} = \left(a_k - i\frac{b_k\sqrt{1+\alpha k^2}}{k^2}\right) \\
&\quad \cdot k\sin\left(\frac{k(\eta+\xi)}{2}\right)\sin\left(\frac{k(\eta-\xi)}{2}\right), \quad k\in\mathbb{N}^*, \\
&x_0 = 0,
\end{aligned}$$

As $\lim_{|k|\to\infty}|\nu_{k+1}-\nu_k| = 1/\sqrt{\alpha}$, then for any real $T > 2\pi\sqrt{\alpha}$, there exists a constant $C_1 > 0$ such that,

$$\begin{aligned}(4.5)\quad \int_0^T |\phi_x(\eta,t)-\phi_x(\xi,t)|^2\mathrm{d}t &\\ \geq C_1\sum_{k\geq 1}k^2\left(a_k^2+\frac{b_k^2(1+\alpha k^2)}{k^4}\right)&\\ \cdot\left[\sin\left(\frac{k(\eta+\xi)}{2}\right)\sin\left(\frac{k(\eta-\xi)}{2}\right)\right]^2.&\end{aligned}$$

When $\frac{\eta+\xi}{2\pi}$ and $\frac{\eta-\xi}{2\pi}$ belong to A, it follows from (2.1) that there exists a constant $C > 0$ such that for all $k \geq 1$,

$$\begin{aligned}(4.6)\quad \left|\sin\left(\frac{k(\eta\pm\xi)}{2}\right)\right| &= \left|\sin\left\{\pi\left[\frac{k(\eta\pm\xi)}{2\pi}-p\right]\right\}\right|\\ &\geq \left|\sin\left(\frac{\pi C}{k}\right)\right| \geq \frac{C}{k}.\end{aligned}$$

Inequalities (4.5) and (4.6) imply that

$$\int_0^T|\phi_x(\eta,t)-\phi_x(\xi,t)|^2\mathrm{d}t \geq c\sum_{k\geq1}(a_k^2k^{-2}+b_k^2k^{-4}),$$

which is exactly (4.3) for $\beta = 1$. This fact completes the proof of the first part of Theorem 1.1.

When $\frac{\eta+\xi}{2\pi}$ and $\frac{\eta-\xi}{2\pi}$ belong to B_ε, it follows from (2.2) that there exists a constant $C > 0$ such that for all $k \geq 1$,

$$(4.7)\qquad \left|\sin\left(\frac{k(\eta\pm\xi)}{2}\right)\right| \geq \frac{C}{k^{1+\varepsilon}}.$$

Inequalities (4.5) and (4.7) imply that

$$\int_0^T|\phi_x(\eta,t)-\phi_x(\xi,t)|^2\mathrm{d}t \geq c\sum_{k\geq1}(a_k^2k^{-2-4\varepsilon}+b_k^2k^{-4-4\varepsilon}),$$

which is exactly (4.3) when $\beta = 1+2\varepsilon$. This fact completes the proof of the second part of Theorem 1.1.

4.2 The lack of controllability in short control time (Proof of Theorem 1.2) We prove the lack of L^2-controllability when $0 < T < 2\pi\sqrt{\alpha}$ in this section. Let $0 < T < 2\pi\sqrt{\alpha}$ and ξ, η in $(0,\pi)$ be arbitrary.

From Proposition 4.1, for any $\varepsilon \geq 0$, we aim to find $\{(\phi_m^0,\phi_m^1)\}_{m\in\mathbb{N}^*}$ such that

$$\int_0^T|\phi_{m,x}(\eta,t)-\phi_{m,x}(\xi,t)|^2\mathrm{d}t \to 0, \quad \text{as } m\to\infty$$

and

$$\|\phi_m^0\|_{H^{1-\varepsilon}}^2+\|\phi_m^1\|_{H^{-\varepsilon}}^2 \geq c > 0$$

for any $m \geq 1$. As in Section 4.1, we denote

$$(4.8)\qquad \lambda_n = -\lambda_{-n} = \frac{n^2}{\sqrt{1+\alpha n^2}}, \quad n\in\mathbb{N}^*.$$

Obviously, $\{\lambda_n\}_{n\in\mathbb{Z}^*}$ is a strictly increasing sequence and $\lim_{|n|\to\infty}|\lambda_{n+1}-\lambda_n| = 1/\sqrt{\alpha} > 0$. Adding or subtracting finite numbers in the sequence does not affect the result of Proposition 2.3, so we can apply Proposition 2.3 to the sequence $\{\lambda_n\}_{n\in\mathbb{Z}^*}$. Define $N(x,r)$ as in Proposition 2.3 corresponding to $\{\lambda_n\}_{n\in\mathbb{Z}^*}$. We have the following lemma.

Lemma 4.2. *Let $\{\lambda_n\}_{n\in\mathbb{Z}^*}$ and $N(x,r)$ be defined above. We have that*

$$(4.9)\qquad \frac{N(x,r)}{r}\to\sqrt{\alpha}, \quad \text{as } r\to\infty$$

holds uniformly relative to x in $\mathbb{R}$.

Here we skip the proof of this lemma, which can be found in the full version of the paper (see [3]). As $0 < T < 2\pi\sqrt{\alpha}$, we can choose T' such that $0 < T < T' < 2\pi\sqrt{\alpha}$. Let f in $L^2(0,2\pi\sqrt{\alpha})$ be a real valued function such that $f(t) = 0$ if $0 \leq t \leq T$ and $\|f\|_{L^2(0,T')} \neq 0$. According to Lemma 4.2 and Proposition 2.3, the family $\{\mathrm{e}^{i\lambda_n t}\}_{n\in\mathbb{Z}^*}$ contains a subfamily $\{\mathrm{e}^{i\lambda_{q_n}t}\}_{n\in\mathbb{Z}^*}$ which forms a Riesz basis in $L^2(0,T')$. Moreover, the subsequence $\{\lambda_{q_n}\}_{n\in\mathbb{Z}}$ satisfies $\lambda_{q_n} = -\lambda_{q_{-n}}$. Then for the function f in $L^2(0,T')$ defined above, there exists a sequence $\{l_n\}_{n\in\mathbb{Z}^*}$ in $l^2(\mathbb{C})$ such that

$$f(t) = \sum_{n\in\mathbb{Z}^*} l_n\mathrm{e}^{i\lambda_{q_n}t} \quad \text{in } L^2(0,T')$$

and

$$0 < \sum_{n\in\mathbb{Z}^*}|l_n|^2 < \infty.$$

Since $f(t)$ is a real valued function, we have $l_n = \overline{l_{-n}}$. Now we can define the sequence $\{(\phi_m^0,\phi_m^1)\}_{m\in\mathbb{N}^*}$ of initial data as the following,

$$\begin{aligned}(4.10)\quad \phi_m^0(x) &= 2\sum_{n=1}^m \mathrm{Re}(l_n)\left[q_n\sin\left(q_n\frac{\eta+\xi}{2}\right)\right.\\ &\quad\left.\cdot\sin\left(q_n\frac{\eta-\xi}{2}\right)\right]^{-1}\sin(q_nx),\\ \phi_m^1(x) &= -2\sum_{n=1}^m \mathrm{Im}(l_n)\left[q_n\sin\left(q_n\frac{\eta+\xi}{2}\right)\right.\\ &\quad\left.\cdot\sin\left(q_n\frac{\eta-\xi}{2}\right)\right]^{-1}\frac{q_n^2}{\sqrt{1+\alpha q_n^2}}\sin(q_nx).\end{aligned}$$

From (4.4), we can see that

$$\frac{\eta-\xi}{2\pi},\ \frac{\eta+\xi}{2\pi}\in\mathbb{R}\setminus\mathbb{Q}$$

is necessary for controllability. Consequently the sequence $\{(\phi_m^0, \phi_m^1)\}_{m\in\mathbb{N}}$ of initial data is well-defined.

Since $0 < \sum_{n\in\mathbb{Z}^*} |l_n|^2 < \infty$ and $l_n = \overline{l_{-n}}$, there exists $m_0 \geq 1$ such that $l_{m_0} \neq 0$. So for $m \geq m_0$ and for any $\varepsilon \geq 0$,

$$\begin{aligned} \|\phi_m^0\|^2_{H^{1-\varepsilon}} &+ \|\phi_m^1\|^2_{H^{-\varepsilon}} \\ &\geq \|\phi_{m_0}^0\|^2_{H^{1-\varepsilon}} + \|\phi_{m_0}^1\|^2_{H^{-\varepsilon}} = c > 0. \end{aligned} \tag{4.11}$$

Moreover, thanks to (4.4), we have

$$\begin{aligned} \int_0^T & |\phi_{m,x}(\eta,t) - \phi_{m,x}(\xi,t)|^2 \mathrm{d}t \\ &= 4\int_0^T \left| \sum_{n=-m, n\neq 0}^{m} l_n \mathrm{e}^{\lambda_{q_n} t} \right|^2 \mathrm{d}t. \end{aligned} \tag{4.12}$$

Since $0 = f(t) = \sum_{n\in\mathbb{Z}^*} l_n \mathrm{e}^{\lambda_{q_n} t}$ in $L^2(0,T)$, we obtain that

$$\int_0^T |\phi_{m,x}(\eta,t) - \phi_{m,x}(\xi,t)|^2 \mathrm{d}t \to 0, \quad \text{as } m \to \infty. \tag{4.13}$$

Relations (4.11) and (4.13) finish the proof of the lack of exact L^2-controllability for any $\varepsilon \geq 0$.

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