Chin. Ann. Math. Ser. B
47(5), 2026, 1–10
DOI: 10.1007/s11401-026-0001-3

Chinese Annals of Mathematics, Series B

Boundary Control and Stabilization of the Saint-Venant Equations*

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Abstract Saint-Venant equations, also known as shallow-water equations, have been used for over a century to model physical phenomena such as navigable rivers or the atmosphere. While quite simple to state, their richness gives rise to many interesting phenomena. This made them a well-studied system. In the last 30 years, the stabilization of the Saint-Venant equations have known a large interest, motivated by their ability to model practical physical systems such as navigable rivers and irrigation channels. These 30 years have been marked by the pioneering work of Coron, largely influential in the field. This paper is a survey in which we present the state of the field, some of the most up-to-date results, as well as new perspectives in this field of research and several interesting open questions that remain.

À Jean-Michel, notre maitre, mentor, et ami.

Keywords
2020 MR Subject Classification

1 Introduction

Introduced in 1871 by Barré de Saint-Venant [7, 9, 8] the *Saint-Venant equations*, also known as *shallow water equations* are among the most renowned systems in fluid mechanics. As all *density-velocity* systems, they consist in two equations: a conservation of mass and a balance of momentum and can be seen as an approximation of Navier-Stokes equations when the height of the fluid is small compared to the characteristic length of the spatial variable. This is the case, for instance, in navigable rivers or in the atmosphere. In their most general form, they are written:

$$\begin{aligned}
&\partial_t A + \partial_x (AV) = 0,\\
&\partial_t (AV) + \partial_x (AV^2) + gA(\partial_x H - S_b(x) + S_f(A, V, x)) = 0,
\end{aligned} \tag{1.1}$$

where A is the wet section of the flow, V is the average horizontal velocity, and g is the gravitational constant, S_b is the influence of the slope, S_f a source term taking into account the friction, and S_P^{-1} is the inverse of the section profile such that $H(t,x) := S_P^{-1}(A(t,x),x)$ is the height of the water. The section profile S_P is a continuous and strictly increasing function with its first variable (which simply means that the wet section increase with the height of the water) which explains that we can talk of S_P^{-1}. In addition $S_P(0) = 0$ (which simply means

Manuscript received (Month) (Date), (Year). Revised (Month) (Date), (Year).
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that when there is no water, well there is no water). In all the following we will consider that $S_b \in C^2([0,L])$ and $S_P \in C^2(\mathbb{R}\times[0,L])$. It is interesting to note that, while they can be derived from Navier-Stokes equations [41], Saint-Venant equations in their original form are hyperbolic and only involve a first order space derivative. In some sense, they resembles the Euler equations more closely. In fact, higher-order approximations of Navier-Stokes equations give rise to a viscosity term (see (3.12)), as shown in [41].

Many studies have been dedicated to the well-posedness of Saint-Venant equations and/or the convergence of the viscous model to the inviscid model (see for instance [22]). In addition to their mathematical richness, Saint-Venant equations are also very valuable for applications, in particular for their ability to model navigable rivers. Navigable rivers are used around the world for fluvial transport or electricity production, and their management is a paramount question both for economical purpose and safety of riverside cities (flood can quickly have a catastrophic impact on rivers' surroundings). To do so, a common practice is to use the river gates and hydraulic installations as means of control (see for instance [11, 88, 38]). From a mathematical point of view this result in a system with boundary controls, typically of the form

$$\begin{aligned} A(t,0)V(t,0) &= \mathcal{B}_1(A(t,0)) \\ A(t,0)V(t,L) &= \mathcal{B}_2(A(t,L)), \end{aligned} \tag{1.2}$$

where B_1 and B_2 are control functions to be chosen. From that point, many useful (control) goals can be considered either from a mathematical or a practical point of view: controllability and stability of the system, stabilization of a target steady-state, output tracking to switch from one target height of the water to another, etc. [88, 89, 38, 78, 31]. In this paper we focus on the stabilization problem:

Problem 1.1 *Given a steady-state* (A^*,V^*) *of* (1.1)*, is it possible to find the conditions on* $\mathcal{B}_1$*,* $\mathcal{B}_2$*,* S_f*,* S_b *and* S_P *such that the system* (1.1), (1.2) *is exponentially stable for a given norm* $\mathcal{X}$ *?*

To be more specific, let us recall the definition of exponential stability

Definition 1.1 *The steady-state* (A^*,V^*) *of* (1.1), (1.2) *is (locally) exponentially stable for the the* $\mathcal{X}$ *norm, where* $\mathcal{X}$ *is a Banach space, if there exists* $\eta>0$*,* $\gamma>0$ *and* $C>0$ *such that, for any* $T>0$ *and any initial condition* (A_0,V_0) *satisfying* $\|A_0-A^*,V_0-V^*\|_{\mathcal{X}}\leq\eta$*, there exists a unique solution* $(A,V)\in C^0([0,+\infty);\mathcal{X})$ *to* (1.1), (1.2) *and one has*

$$\|A(t)-A^*,V(t)-V^*\|_{\mathcal{X}}\leq Ce^{-\gamma t}\|A_0-A^*,V_0-V^*\|_{\mathcal{X}},\;\;\forall t\in[0,+\infty). \tag{1.3}$$

Remark 1.1 Note that this definition of exponential stability depends on $\mathcal{X}$. This is classical for nonlinear infinite dimensional system where the stability in different norms are not equivalent [28].

Remark 1.2 Definition 1.1 is local in the sense that it assumes that the initial conditions are small perturbations of the steady-state. The motivation for such a definition is the fact that for a nonlinear hyperbolic system in general it is not possible to guarantee a global exponential stability in strong norms with boundary controls. Indeed, due to the finite propagation speed and the minimal time needed for one boundary to influence a given point in the domain, if one starts sufficiently close to a shock, then the shock will happen whatever the boundary control.

Remark 1.3 Without loss of generality we can assume that $V^* \geq 0$ and we will assume this in the rest of the paper.

Note that Definition 1.1 only makes sense if the steady-state is subcritical, that is

$$g\partial_A S_P(A^*, \cdot)A^* - V^{*2} > 0. \tag{1.4}$$

Indeed, this condition guarantees that the system has two propagating velocity with opposite sign and, otherwise, the system cannot be well-posed with a condition of the form (1.2). For this reason we introduce the following notation.

Definition 1.2 (Exponential stability) *The system* (1.1), (1.2) *is called exponentially stable for the $\mathcal{X}$ norm if any steady-state* (A^*, V^*) *satisfying the subcritical condition* (1.4) *is exponentially stable for the $\mathcal{X}$ norm.*

In most existing works (see for instance [11, 12, 39, 62]), the system (1.1) is studied with either $S_P = \mathrm{Id}$ such that[1] $A = H$ and a particular choice of source term S_f. While it is agreed upon that a friction source term should satisfy

$$\partial_V S_f \geq 0,\ \partial_H S_f \leq 0, \tag{1.5}$$

given that the friction should increase with the velocity and decrease with the wet area (and therefore the height given that S_P is strictly increasing), there is often a debate on which friction model to choose. Most common sources terms are[2] (see for instance [20, Section 4.5] or [12, 41])

$$S_f(H,V) = \frac{kV|V|}{H},\ S_f(H,V) = kV|V|/H^{4/3},\ S_f(H,V) = kV. \tag{1.6}$$

We will see later that this choice may have some consequences on the stability. However, it was shown recently in [62] that one can still find generic results on the stability independently of the source term, simply under the assumption (1.5)[3].

While the control (1.2) are the most studied, other boundary controls are possible. One type, in particular, is very much used in practice: the *proportional-integral* (PI) control. The interest of this control is that it is robust to a constant error in the control, thanks to its integral part [3, 4, 1]. This control is studied in Section 3.1. The control and stabilization of the Saint-Venant equations has also been studied with yet other types of boundary controls either using the knowledge of the full-state or observers. We will not detail them in this article but one can look at [34] and the references therein. Other results have been obtained for hyperbolic systems in general with controls (1.1) and are not specific to the Saint-Venant equations. Among them, we would like to emphasize the pioneering methods of Coron and Bastin [11, Chapter 6], [23, 10], but also others such as [29, 66, 55, 56, 82, 5].

In the general case of (1.1), (1.2) the most up-to-date results are

[1] a common essentially equivalent choice is $A = bH$ with b a positive constant, see for instance [51, 84]

[2] We do not include here the trivial source term $S(H,V) = 0$ which is also very much studied.

[3] In fact the system and the assumptions considered in [63] are slightly more general than (1.5) but this generality is not needed for the Saint-Venant equations

Theorem 1.1 ([63]) *Any steady-state (A^*, V^*) of the general Saint-Venant equations* (1.1), (1.2) *with* (1.5) *and satisfying* (1.4) *is exponentially stable for the H^p norm for $p \geq 2$ provided that*

$$\begin{aligned} \mathcal{B}_1'(A^*(0)) &\in \left[V^*(0) - \frac{gA^*(0)\partial_A S_P(A^*(0),0)}{V^*(0)}, 0\right], \\ \mathcal{B}_2'(A^*(L)) &\in \mathbb{R} \setminus \left[V^*(L) - \frac{gA^*(L)\partial_A S_P(A^*(L),L)}{V^*(L)}, 0\right]. \end{aligned} \tag{1.7}$$

However, as we will see in the following, this theorem does not allow to recover all the results existing in particular cases when S_f, S_P and S_b follow specific expressions, showing that there is a margin for improvement. In Sections 2.1 to 3.5 we will review the most up to date results in different cases, that have been studied in the last 25 years. We will also prove a new result in Proposition 2.2 and review several interesting open questions. Note that Theorem 1.1 and several results presented in this paper are actually valid for density-velocity systems in general [12, 63] and transfer, in particular, to other well-known systems such as the isentropic or isothermal Euler equations (studied for instance in [35, 36, 52, 54, 53]). Finally, in Section 3 we review new perspective in the stabilization of Saint-Venant equations that have emerged in the recent years beyond the study of the system (1.1), (1.2).

Let us finally note that the most advanced results, as well as most of the results we will present below, rely on finding a so-called basic Lyapunov function, that is a Lyapunov function that can be written as a weighted graph norm of the state, with weights that are non-necessarily constant or uniform. This approach is arguably due to Coron and Bastin, in particular with [25, 10, 11]. In the case of the L^2 norms, these Lyapunov functions have the form

$$V(H, V) = \int_0^L (H, V)^\top Q(H, V)\,dx, \tag{1.8}$$

where Q is a matrix-valued function that values in the space of positive definite matrices of dimension 2. An expression in the general case for H^p and C^q norm can be found for instance in [57, Section 3].

2 Stabilization of the Saint-Venant equations: specific cases

In this section, we will assume that $S_P = \mathrm{Id}$ such that the Saint-Venant system become

$$\begin{aligned} &\partial_t H + \partial_x(HV) = 0, \\ &\partial_t(HV) + \partial_x(HV^2) + gH(\partial_x H - S_b(x) + S_f(H, V, x)) = 0, \end{aligned} \tag{2.1}$$

and, except stated otherwise, we will assume that the boundary conditions are

$$\begin{aligned} H(t,0)V(t,0) &= G_0(H(t,0)) \\ H(t,L)V(t,L) &= G_1(H(t,L)), \end{aligned} \tag{2.2}$$

Denote

$$b_0 = \frac{G_0'(H^*(0)) - V^*(0)}{H^*(0)}, \quad b_1 = \frac{G_1'(H^*(L)) - V^*(L)}{H^*(L)}. \tag{2.3}$$

The stability depends crucially on these two parameters of the boundary conditions.

2.1 The homogeneous case $S_f(H, V) = S_b = 0$

The homogeneous case was first studied by Li and Greenberg in 1984 in [42], in a more general setting using a careful study of the characteristics, and then specifically to the Saint-Venant system by Coron, d'Andréa-Novel and Bastin in [24] and [25] using a Lyapunov approach equivalent to the H^2 norm and then later to the H^p norm, $p \geq 2$ (see Bastin Coron [11]) and the C^q norm, $q \geq 1$ (see [23]). What is shown is the following

Theorem 2.1 ([42, 24, 25, 11, 23]) *Assume that*

$$\left| \frac{b_0 + \sqrt{g/H^*}}{b_0 - \sqrt{g/H^*}} \cdot \frac{b_1 - \sqrt{g/H^*}}{b_1 + \sqrt{g/H^*}} \right| < 1, \tag{2.4}$$

then any steady-state (H^*, V^*) *of the system* (2.1), (2.2) *satisfying* (1.4) *is exponentially stable for the* H^p*-norm and the* C^q *norm for any* $p \geq 2$, $q \geq 1$.

Remark 2.1 The condition (2.4) is essentially necessary and sufficient for the exponential stability in H^p norms and in the C^q norms (see [11, Theorem 2.4] which deals with the linear system but the approach can be directly extended to the nonlinear system, see also [89] and Appendix A for the current case). We give such a proof in Appendix A, i.e. we show that the system (2.1), (2.2) is not exponentially stable if

$$\left| \frac{b_0 + \sqrt{g/H^*}}{b_0 - \sqrt{g/H^*}} \cdot \frac{b_1 - \sqrt{g/H^*}}{b_1 + \sqrt{g/H^*}} \right| > 1. \tag{2.5}$$

This result can be generalized to density-velocity systems in general.

In the particular case where the system is inhomogeneous and $S_f(H, V) \neq 0$ and $S_b \neq 0$, then one can still stabilize any constant steady-state [14][4].

Theorem 2.2 ([14]) *Any constant steady-state* (H^*, V^*) *of* (2.1), (2.2) *satisfying* (1.4) *is exponentially stable for the* H^2 *norm, provided that* (2.4) *holds.*

2.2 The case $S_f(H, V) = k|V|V/H$

This case is arguably one of the most studied (if not the most studied in mathematical studies) and corresponds to the Chézy model [11, 20, 21]. Because the Saint-Venant equations are a 2×2 system, it is useful, when willing to use a basic quadratic Lyapunov function, to consider the characterization given in [10] which, in this case, become

Theorem 2.3 ([10, 11]) *Let* (H^*, V^*) *be a steady-state satisfying* (1.4), *there exists* G_0 *and* G_1 *such that there exists a basic quadratic Lyapunov function for the* H^2 *norm for the system* (2.1), (2.2) *if and only if there exists a solution* η *to the equation*

$$\eta' = \left| \frac{a}{\lambda_1} + \frac{b}{\lambda_2} \eta^2 \right|, \tag{2.6}$$

[4] Note that [14] is actually dealing with the specific case $S_f(H, V) = kV^2/H$, but their argument can be extended to other cases

where

$$
\begin{aligned}
&\lambda_1(x)=\sqrt{gH^*}+V^*,\quad \lambda_2(x)=\sqrt{gH^*}-V^*,\quad a(x)=\varphi(x)\delta_1(x),\quad b(x)=\varphi^{-1}(x)\gamma_2(x),\\
&\varphi(x)=\frac{\varphi_1(x)}{\varphi_2(x)},\quad \varphi_1(x)=\exp\left(\int_0^x\frac{\gamma_1(s)}{\lambda_1(s)}ds\right),\quad \varphi_2(x)=\exp\left(-\int_0^x\frac{\delta_2(s)}{\lambda_2(s)}ds\right),\\
&\gamma_1(x)=-\frac{3f(H^*,V^*)}{4(\sqrt{gH^*}+V^*)}+\frac{kgV^*}{H^*}-\frac{kgV^{*2}}{2H^*\sqrt{gH^*}},\\
&\delta_1(x)=-\frac{f(H^*,V^*)}{4(\sqrt{gH^*}+V^*)}+\frac{kgV^*}{H^*}+\frac{kgV^{*2}}{2H^*\sqrt{gH^*}},\\
&\gamma_2(x)=\frac{f(H^*,V^*)}{4(\sqrt{gH^*}-V^*)}+\frac{kgV^*}{H^*}-\frac{kgV^{*2}}{2H^*\sqrt{gH^*}},\\
&\delta_2(x)=\frac{3f(H^*,V^*)}{4(\sqrt{gH^*}-V^*)}+\frac{kgV^*}{H^*}+\frac{kgV^{*2}}{2H^*\sqrt{gH^*}},\\
&f(H^*,V^*)=\frac{kgV^{*2}}{H^*}-gS_b.
\end{aligned}
\tag{2.7}
$$

In practice, from any function $\eta\in C^1([0,L])$ such that $\eta(0)>0$ and

$$\eta'>\left|\frac{a}{\lambda_1}+\frac{b}{\lambda_2}\eta^2\right|,\tag{2.8}$$

one can build a quadratic Lyapunov function and the condition on G_0 (resp. G_1) such that (H^*,V^*) is exponentially stable are directly linked to $\eta(0)$ (resp. $\eta(L)$). This reduces the search of sufficient conditions on G_0 and G_1 for the stability of the Saint-Venant equations (2.1), (2.2) to the search of a function $\eta>0$ satisfying (2.8). In 2017, Bastin and Coron constructed a function η inspired from the linearized version of the physical mechanical energy of the system (given by $E_m=gH^2+HV^2$) which allowed them to show the following result when there is no slope:

Theorem 2.4 ([12]) *Assume that $S_b=0$. A steady-state (H^*,V^*) of* (2.1), (2.2) *satisfying* (1.4) *is exponentially stable for the H^2 norm if*

$$
\begin{aligned}
&b_0\in\left[-\frac{g}{V^*(0)}-g\sqrt{\frac{1}{V^*(0)^2}-\frac{1}{gH^*(0)}},-\frac{g}{V^*(0)}+g\sqrt{\frac{1}{V^*(0)^2}-\frac{1}{gH^*(0)}}\right],\\
&b_1\in\mathbb{R}\setminus\left(-\frac{g}{V^*(L)}-g\sqrt{\frac{1}{V^*(L)^2}-\frac{1}{gH^*(L)}},-\frac{g}{V^*(L)}+g\sqrt{\frac{1}{V^*(L)^2}-\frac{1}{gH^*(L)}}\right).
\end{aligned}
\tag{2.9}
$$

The function η constructed in [12] is a strict super-solution of (2.6). The exact solution of (2.6) was later found in [62] where it was observed, interestingly, that this solution has a fully explicit form for the Saint-Venant equations. The Lyapunov function thus resulting, optimal in some sense, was used to prove the following result

Theorem 2.5 ([62]) *A steady-state (H^*,V^*) of* (2.1), (2.2) *satisfying* (1.4) *is exponentially stable for the H^2 norm if*

$$b_0\in\Big[-\frac{g}{V^*(0)},-\frac{V^*(0)}{H^*(0)}\Big],\quad b_1\in\mathbb{R}\setminus\left[-\frac{g}{V^*(L)},-\frac{V^*(L)}{H^*(L)}\right].\tag{2.10}$$

One can note that, in this theorem, the assumption $S_b = 0$ is not needed and this result holds whatever the slope. In fact, this theorem has two particularities:

- it can be generalized to a larger class of systems: *the density-velocity systems* (which includes the general Saint-Venant equations (1.1), (1.2), see [63])
- the result holds for any $L > 0$. This is in contrast with the no-slope case as, when $S_f(H^*, V^*) - S_b > 0$, the steady-state cease to exist after a finite length $L > 0$ (see for instance [50]) which therefore gives a bound on L.

In fact, the existence of sufficient condition on G_0 and G_1 to stabilize the system for any length $L > 0$ is remarkable given that such result is usually impossible in general for hyperbolic systems with a source term. This impossibility in general was observed for instance by Bastin and Coron in [11, Section 5.6] but also by Gugat and Herty in [48] and necessary and sufficient conditions for 2×2 linear systems where found by Huang, Wang and Zhou in [67]. The remarkable fact that this limits does not appear here is the consequence of a special hidden structure in the Saint-Venant equations (and in density-velocity systems in general).

Interestingly, one can note that the sufficient conditions of Theorem 2.5 do not completely contain the sufficient conditions of Theorem 2.4. In fact, another interesting (and explicit) function η was found in [60] and the consequent sufficient conditions of stabilization are

Theorem 2.6 ([60]) *Assume that $S_b = 0$. A steady-state (H^*, V^*) of* (2.1), (2.2) *satisfying* (1.4) *is exponentially stable for the H^2 norm if*

$$b_0 \in (-\infty, 0], \quad b_1 \in \mathbb{R} \setminus \left[-\sqrt{\frac{g}{H^*(L)}} \frac{\lambda_1(L) + m\lambda_2(L)}{\lambda_1(L) - m\lambda_2(L)}, -\sqrt{\frac{g}{H^*(L)}} \frac{\lambda_1(L) - m\lambda_2(L)}{\lambda_1(L) + m\lambda_2(L)} \right], \tag{2.11}$$

where λ_1, λ_2 are defined by (2.7), *and*

$$m = \frac{H^*(L)^{\frac{5}{2}} + \frac{\sqrt{g}}{2V^*(L)H^*(L)}(H^*(L)^4 + H^*(0)^4) + \frac{V^*(L)H^*(L)}{\sqrt{g}}(H^*(L) - H^*(0))}{-H^*(L)^{\frac{5}{2}} + \frac{\sqrt{g}}{2V^*(L)H^*(L)}(H^*(L)^4 + H^*(0)^4) + \frac{V^*(L)H^*(L)}{\sqrt{g}}(H^*(L) - H^*(0))}. \tag{2.12}$$

Again, one can remark that the conditions of Theorem 2.6 only have a partial overlap with the ones of Theorem 2.5 and Theorem 2.4, which means that the most general known conditions for the stabilization of the Saint-Venant equations (2.1), (2.2) to this day are summarized in the following corollary

Theorem 2.7 *Assume that $S_b = 0$. A steady-state (H^*, V^*) of* (2.1), (2.2) *satisfying* (1.4) *is exponentially stable for the H^2 norm if*

$$\begin{aligned}(b_0, b_1) \in (-\infty, 0] \times &\left(\mathbb{R} \setminus [b^-_{1,\mathrm{T.2.6}}, b^+_{1,\mathrm{T.2.6}}]\right) \bigcup [b^-_{0,\mathrm{T.2.4}}, b^+_{0,\mathrm{T.2.4}}] \times \left(\mathbb{R} \setminus (b^-_{1,\mathrm{T.2.4}}, b^+_{1,\mathrm{T.2.4}})\right) \\ &\bigcup [b^-_{0,\mathrm{T.2.5}}, b^+_{0,\mathrm{T.2.5}}] \times \left(\mathbb{R} \setminus [b^-_{1,\mathrm{T.2.5}}, b^+_{1,\mathrm{T.2.5}}]\right),\end{aligned} \tag{2.13}$$

where

$$b^\pm_{1,\mathrm{T.2.6}} = -\sqrt{\frac{g}{H^*(L)}} \frac{\lambda_1(L) \mp m\lambda_2(L)}{\lambda_1(L) \pm m\lambda_2(L)},$$

$$b^\pm_{0,\mathrm{T.2.4}} = -\frac{g}{V^*(0)} \pm g\sqrt{\frac{1}{V^*(0)^2} - \frac{1}{gH^*(0)}}, \quad b^\pm_{1,\mathrm{T.2.4}} = -\frac{g}{V^*(L)} \pm g\sqrt{\frac{1}{V^*(L)^2} - \frac{1}{gH^*(L)}},$$

$$b^-_{0,\mathrm{T}.2.5} = -\frac{g}{V^*(0)}, \quad b^+_{0,\mathrm{T}.2.5} = -\frac{V^*(0)}{H^*(0)}, \quad b^-_{1,\mathrm{T}.2.5} = -\frac{g}{V^*(L)}, \quad b^+_{1,\mathrm{T}.2.5} = -\frac{V^*(L)}{H^*(L)},$$

with λ_1, λ_2 defined by (2.7), *and m defined by* (2.12).

The relatively complex region of stability given by Theorem 2.7 suggests to compare the different bounds to see if any simplification is possible. Let us first compare the regions in Theorems 2.4 to 2.6. Direct computations shows that

$$\begin{aligned} b^-_{0,\mathrm{T}.2.4} < b^-_{0,\mathrm{T}.2.5} < b^+_{0,\mathrm{T}.2.5} < b^+_{0,\mathrm{T}.2.4}, \\ b^-_{1,\mathrm{T}.2.4} < b^-_{1,\mathrm{T}.2.5} < b^+_{1,\mathrm{T}.2.5} < b^+_{1,\mathrm{T}.2.4}. \end{aligned}$$

Using $H^*(0) > H^*(L)$ (recall that $S_b = 0$), direct computation shows that $1 < m < \lambda_1(L)/\lambda_2(L)$ and we thus have

$$b^-_{1,\mathrm{T}.2.6} < b^-_{1,\mathrm{T}.2.5}, \quad b^+_{1,\mathrm{T}.2.6} > b^+_{1,\mathrm{T}.2.5}.$$

Comparing the region of (2.11) and the region of (2.9), we have

$$b^-_{1,\mathrm{T}.2.6} < b^-_{1,\mathrm{T}.2.4}, \quad b^+_{1,\mathrm{T}.2.6} > b^+_{1,\mathrm{T}.2.4},$$

if and only if

$$m > \sqrt{\frac{\lambda_1(L)}{\lambda_2(L)}}.$$

This condition may not always hold, depending on the steady-state and the length of the domain. Indeed, let $H^*(x)V^*(x) = Q^*$ for all $x \in [0, L]$ with some positive constant Q^*. From the equation of steady-state, we have

$$\partial_x H^* = -\frac{gkV^{*2}}{gH^* - V^{*2}} = -\frac{gkQ^{*2}}{gH^{*3} - Q^{*2}}. \tag{2.14}$$

Let $H^*(L) = H^*_L$ with $gH^{*3}_L - Q^{*2} > 0$. Integrating (2.14) over $x \in [0, L]$, we have

$$\frac{g}{4}H^*(0)^4 - Q^{*2}H^*(0) = \frac{g}{4}H^{*4}_L - Q^{*2}H^*_L + gkQ^{*2}L.$$

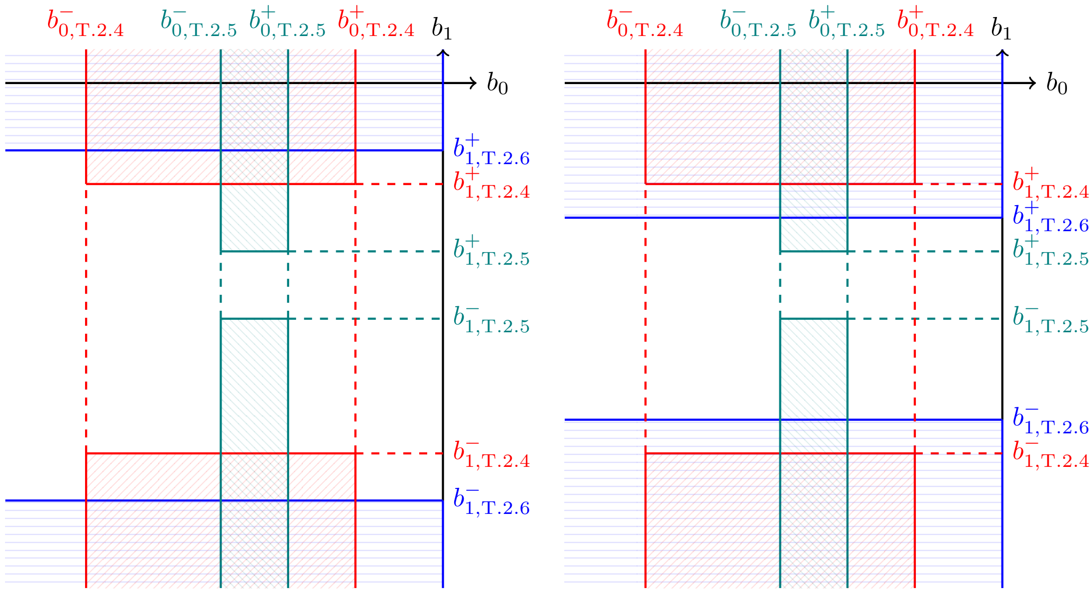
Therefore, we have

$$\begin{cases} H^*(0) \to H^*_L & (\text{resp. } +\infty), \\ m \to \dfrac{\lambda_1(L)}{\lambda_2(L)} & (\text{resp. } 1), \end{cases} \qquad \text{as } L \to 0 \quad (\text{resp. } +\infty).$$

Thus, for given $Q^* > 0$ and $H^*_L > (Q^{*2}/g)^{1/3}$, there exists $L_0 > 0$ such that

$$\begin{cases} m > \sqrt{\dfrac{\lambda_1(L)}{\lambda_2(L)}}, & \text{if } L < L_0, \\ m < \sqrt{\dfrac{\lambda_1(L)}{\lambda_2(L)}}, & \text{if } L > L_0. \end{cases}$$

The comparison among the regions in Theorems 2.4 to 2.6 is shown in Figure 1.



(a) $m > \sqrt{\frac{\lambda_1(L)}{\lambda_2(L)}}$, namely, $L < L_0$.

(b) $m < \sqrt{\frac{\lambda_1(L)}{\lambda_2(L)}}$, namely, $L > L_0$.

Figure 1: Regions in Theorems 2.4 to 2.6. The regions for Theorems 2.4, 2.5 and 2.6 are represented in red, green and blue, respectively.

The peculiar form the this condition suggests that, despite nearly 30 years of previous work on this question, optimal conditions on G_0 and G_1 to stabilize (2.1), (2.2) are yet to be found, even only in the case where $S_b = 0$. This is a major open question that can be divided (at least) in two:

- Given (H^*, V^*), is it possible to find explicit necessary and sufficient conditions on G_0 and G_1 for the existence of a basic Lyapunov function for the H^2 norm?
- Given (H^*, V^*), is it possible to find explicit necessary and sufficient conditions on G_0 and G_1 for (H^*, V^*) to be exponentially stable for the H^2 norm?

The second one is obviously harder (since the existence of a basic Lyapunov function implies the exponential stability). Note, however, that when $S_b = S_f = 0$ this question is answered by Theorem 2.1 and Remark 2.1. The comparison among the regions in Theorems 2.1 and 2.4 to 2.6 is shown in Figure 2.

Another open question: H^2 vs C^1 norm Note that the conditions (2.10) allow the existence of a basic Lyapunov function for the H^2 norm (and in fact the H^p norm for any $p \geq 2$) but not the existence of a basic Lyapunov function for the C^1 norm (or C^q norm, $q \geq 1$). Indeed, there exists (H^*, V^*) satisfying (1.4) and G_0, G_1 satisfying (2.10) such that there does not exist a basic Lyapunov function for the C^1 norm. The same holds in the general case with the system (2.1), (2.2) and the condition (1.7) of Theorem 1.1. This begs for the following open question

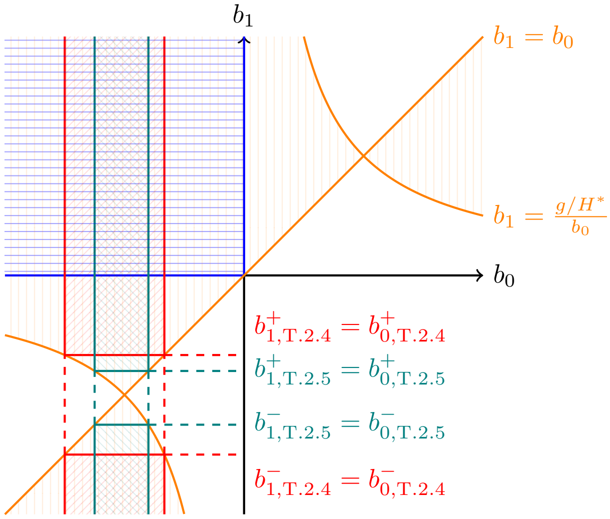


Figure 2: Regions in Theorems 2.1 and 2.4 to 2.6 for the case $S_f = S_b = 0$. The regions for Theorems 2.1, 2.4, 2.5 and 2.6 are represented in orange, red, green and blue, respectively.

- Assume that (H^*, V^*) satisfy (1.4) and G_0, G_1 satisfy (2.10), is the steady-state (H^*, V^*) of (2.1), (2.2) exponentially stable for the C^1 norm?

The same question can be asked in the general case with (A^*, V^*) a steady-state of (1.1), (1.2) and G_0, G_1 satisfying (1.7) (with likely the same answer).

2.3 The case $S_f(H,V) = kV/H^\alpha$

The case $S_f(H,V) = kV/H^\alpha$ with $\alpha \geq 0$[5]. is an important case. It was used, for instance, in [41, Proposition 4.1] with $\alpha = 1$ and in [22, Section 6] with $\alpha = 0$. In this case, a remarkable behavior detailed below when there is a constant slope.

First note that when there is a constant slope $S_b > 0$ then for any Q^* there exists a unique constant steady-state (H^*, V^*) given by

$$H^* = Q^*/V^*, \quad V^* = (S_b Q^{*\alpha}/k)^{\frac{1}{1+\alpha}}. \tag{2.15}$$

Since classical smooth solutions to the Saint-Venant equations may fail to exist globally when the initial data are not sufficiently small, discontinuities and shock waves can develop in finite time. Therefore, the framework of entropy solutions provides a natural setting for selecting physically admissible weak solutions and studying the well-posedness and stability properties of the controlled system.

[5]The case $\alpha < 0$ is usually considered non-physical as S_f represents the friction from the bottom whose influence is non-increasing with the size of the fluid layer

In this case, we will consider the entropic solution of (2.1), (2.2) with initial condition $(H_0, V_0) \in BV(0,L)$ satisfying $H_0 \geq 0$ that is subcritical near the end of the channel, that is $gH_0(L) > V_0^2(L)$. Let $u = (H,Q)$ and set

$$F(u) = \begin{pmatrix} Q \\ \frac{Q^2}{H} + \frac{gH^2}{2} \end{pmatrix} \quad \mathcal{S}(u) = \begin{pmatrix} 0 \\ -gH(S_f(H, \frac{Q}{H}) - S_b) \end{pmatrix}. \tag{2.16}$$

We introduce the following notion of entropy pair

Definition 2.1 (Entropy pairs [22]) *A pair of mapping $(\eta, q) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^2$ is called an entropy pair of system (2.1) if the pair satisfies the 2×2 hyperbolic system:*

$$\nabla q(u) = \nabla \eta(u) \nabla F(u). \tag{2.17}$$

Furthermore, $\eta(H,Q)$ is called a weak entropy if it is defined on $(0,+\infty) \times \mathbb{R}$ instead of $\mathbb{R}_+ \times \mathbb{R}$ and satisfies

$$\lim_{h\to 0} \eta(h, hc) = 0, \quad \forall c \in \mathbb{R}. \tag{2.18}$$

Moreover, if $\eta \in C^2((0,+\infty) \times \mathbb{R}; \mathbb{R})$ is a weak-entropy, it is said convex if $\nabla^2 \eta$ is non-negative definite on $(0,+\infty) \times \mathbb{R}$.

We define the following notion of entropic solutions following [22].

Definition 2.2 (Entropic solutions) *A pair of functions $u = (H,Q)$ is called a entropic solution of (2.1), (2.2) with initial condition $(H_0, Q_0) \in BV(0,L)$ satisfying $H_0 \geq 0$ provided that*

(i) $(H,Q) \in C^0([0,T]; L^1(0,L)) \cap L^\infty((0,T); BV(0,L))$*;*

(ii) For any convex weak entropy pair (η, q), the following entropy inequality holds in the distributional sense:

$$\partial_t \eta(u) + \partial_x q(u) \leq \nabla \eta(u) \cdot \mathcal{S}(u), \tag{2.19}$$

namely, for any non-negative test function $\psi \in C_c^1([0,T); C^1[0,L])$,

$$\begin{aligned} &\iint_{(0,T)\times(0,L)} [\eta(u)\partial_t \psi + q(u)\partial_x \psi + \nabla \eta(u) \cdot \mathcal{S}(u)\psi] dx dt \\ &+ \int_0^T [q(u(t,0+))\psi(t,0) - q(u(t,L-))\psi(t,L)] dt + \int_0^L \eta(u(0,x))\psi(0,x) dx \geq 0; \end{aligned} \tag{2.20}$$

In the above conditions, $\frac{Q^2}{H^{1+\alpha}}, \eta(u), q(u)$ are defined by zero on set $\{H = 0\}$.

In this framework, one has the following remarkable property

Proposition 2.1 *Assume that*

$$\begin{aligned} &G_0(H^*) = G_1(H^*) = Q^* \\ &g(s - H^*)(2G_0(s) - V^*(s + H^*)) + \frac{G_0(s)}{s^2}(G_0(s) - sV^*)^2 \leq 0, \quad \forall s \geq 0, \\ &g(s - H^*)(2G_1(s) - V^*(s + H^*)) + \frac{G_1(s)}{s^2}(G_1(s) - sV^*)^2 \geq 0, \quad \forall s \geq 0. \end{aligned} \tag{2.21}$$

Then, let $(H,V)\in C^0([0,T];L^1(0,L))\cap L^\infty((0,T);BV(0,L))$ *be an entropic solution of* (2.1), (2.2) *with initial condition* $(H_0,V_0)\in BV(0,L)$ *and assume that* $H(t,\cdot)\geq 0$ *and* $gH(t,L)>V^2(t,L)$ *then, one has*

$$\begin{aligned}&g\|H(t)-H^*\|_{L^2}^2+\|\sqrt{H(t)}(V(t)-V^*)\|_{L^2}^2+2gk\int_0^t\int_0^L H\left(\frac{V(s)}{H^\alpha(s)}-\frac{V^*}{H^{*\alpha}}\right)(V(s)-V^*)\,dxds\\&\leq g\|H_0-H^*\|_{L^2}^2+\|\sqrt{H_0}(V_0-V^*)\|_{L^2}^2.\end{aligned}\tag{2.22}$$

Remark 2.2 The condition $gH(t,L)>V^2(t,L)$ ensures that one of the propagation velocity is negative at $x=L$ and that consequently the right boundary conditions (2.2) at $x=L$ is well-defined.

Remark 2.3 In particular, if $\alpha=0$ and there exists $T>0$ and positive constants $M,\delta>0$ such that $\delta\leq H\leq M$ for any $(t,x)\in[0,T]\times[0,L]$ then there exists $c_\delta, C_\delta>0$ such that we have the following L^2 norm estimate, for all $t\in[0,T]$,

$$\|(H,V)-(H^*,V^*)\|_{L^2}\leq C_\delta\|(H_0,V_0)-(H^*,V^*)\|_{L^2}-c_\delta\left(\int_0^t 2gk\|V(t)-V^*(t)\|_{L^2}^2ds\right)^{1/2}.\tag{2.23}$$

Remark 2.4 Let us consider the condition (2.21) locally around H^*. For a smooth function G satisfying $G(H^*)=H^*V^*$, by a Taylor expansion, we have

$$\begin{aligned}&g(s-H^*)(2G(s)-V^*(s+H^*))+\frac{G(s)}{s^2}(G(s)-sV^*)^2\\&=\left[g(2G'(H^*)-V^*)+\frac{V^*}{H^*}(G'(H^*)-V^*)^2\right](s-H^*)^2+O((s-H^*)^3).\end{aligned}\tag{2.24}$$

Therefore, for $H(t,0)$ and $H(t,L)$ close to H^*, condition (2.21) can be obtained from

$$\begin{aligned}&g(2G_0'(H^*)-V^*)+\frac{V^*}{H^*}(G_0'(H^*)-V^*)^2<0,\\&g(2G_1'(H^*)-V^*)+\frac{V^*}{H^*}(G_1'(H^*)-V^*)^2>0,\end{aligned}\tag{2.25}$$

which are equivalent to

$$G_0'(H^*)\in(a_-,a_+),\quad G_1'(H^*)\in\mathbb{R}\setminus[a_-,a_+],\tag{2.26}$$

with

$$a_\pm=\frac{1}{V^*}\left(V^{*2}-gH^*\pm\sqrt{gH^*(gH^*-V^{*2})}\right).\tag{2.27}$$

Notice that

$$\frac{a_\pm-V^*}{H^*}=-\frac{g}{V^*}\pm g\sqrt{\frac{1}{V^{*2}}-\frac{1}{gH^*}}.\tag{2.28}$$

Therefore, locally around H^*, condition (2.26) coincides with condition (2.9) from Theorem 2.4 and the result of [12].

Proposition 2.1 and Remark 2.3 are proved later in this section. In fact, we have a stronger result:

Proposition 2.2 *Assume that $\alpha = 0$ and G_0, G_1 satisfy (2.21) and that*

$$G_0(h) = G_1(h) = \frac{S_b}{k} h \text{ has only one non-negative solution } h = H^*. \tag{2.29}$$

Then for any entropic solution $(H, V) \in C^0([0, +\infty); L^1(0, L)) \cap L^\infty((0, +\infty); BV(0, L))$ of (2.1), (2.2) satisfying $H(t, \cdot) \geq 0$ and $gH(t, L) > V^2(t, L)$, this solution converges to H^*, V^* in L^1, namely

$$\lim_{t \to +\infty} \|H(t, \cdot) - H^*, V(t, \cdot) - V^*\|_{L^1} = 0. \tag{2.30}$$

Remark 2.5 One can note that, remarkably, this is a (semi-)global asymptotic stability result: indeed, the initial condition could be far away from the targeted steady-state. This is all the more interesting that, in general, the stabilization results for quasilinear hyperbolic systems are local.

Remark 2.6 It is worth noting that, for entropy solutions, dry states, namely $H = 0$, may occur locally even with a positive initial condition $H_0 > 0$, see [70, Chapter 13] for details.

Proof of Proposition 2.1. The system (2.1) can be written in conservative form as system

$$\partial_t u + \partial_x (F(u)) = \mathcal{S}(u). \tag{2.31}$$

We set the entropy $\eta(u) = \frac{1}{2}\left(\frac{Q^2}{H} + gH^2\right)$. Let $u^* = (H^*, Q^*)^\top$ be the (unique) steady-state of (2.31), we define the relative entropy

$$\eta(u|u^*) = \eta(u) - \eta(u^*) - \nabla\eta(u^*) \cdot (u - u^*). \tag{2.32}$$

One can note that, with the variable $V = Q/H$ one has

$$\begin{aligned} \eta(u|u^*) &= \frac{1}{2} g(H^2 - H^{*2}) + \frac{1}{2}(HV^2 - H^* V^{*2}) - (gH^* - \frac{1}{2} V^{*2})(H - H^*) - V^*(HV - H^* V^*) \\ &= \frac{1}{2}\left(g(H - H^*)^2 + H(V - V^*)^2\right), \end{aligned} \tag{2.33}$$

which is indeed the quantity we expect to study. Let us now define the function

$$W(U) = \int_0^L \eta(U|u^*) dx. \tag{2.34}$$

One can first observe that V is well-defined for any function $U \in L^\infty(0, L)$. Moreover we can also observe that η if one defines

$$q(u) = gHQ + \frac{Q^3}{2H^2}. \tag{2.35}$$

then η is a weak entropy for the system (2.31) associated to the entropy-flux q. Note that η is strictly convex on $\{(H, Q) \mid H > 0\}$. Indeed,

$$\nabla^2 \eta(H, Q) = \begin{pmatrix} g + \dfrac{Q^2}{H^3} & -\dfrac{Q}{H^2} \\ -\dfrac{Q}{H^2} & \dfrac{1}{H} \end{pmatrix}, \qquad \det \nabla^2 \eta(H, Q) = \frac{g}{H} > 0,$$

and $1/H > 0$, therefore $\nabla^2\eta(H,Q)$ is positive definite for all $H > 0$, as a consequence (η, q) is a convex weak entropy pair. Let $u \in L^\infty((0,T); BV(0,L))$ be an entropic solution of (2.1), (2.2). One has, by definition of an entropic solution and in a distributional sense:

$$\partial_t\eta(u) + \partial_x q(u) \leq \nabla\eta(u)\cdot\mathcal{S}(u) \tag{2.36}$$

Since $\partial_Q\eta(u) = Q/H$, one has

$$\nabla\eta(u)\cdot\mathcal{S}(u) = \frac{Q}{H}\,\mathcal{S}_2(u).$$

In particular, in the case $\mathcal{S}_2(u) = -gH(kV/H^\alpha - S_b)$ (i.e. the source term in (2.1) in the case $S_f(H,V) = kV/H^\alpha$),

$$\nabla\eta(u)\cdot\mathcal{S}(u) = -gV\,H(kV/H^\alpha - S_b).$$

and therefore let $a := \nabla\eta(u^*)$, which is a constant vector. Since u is a weak solution of (2.31), we have in $\mathcal{D}'((0,T)\times(0,L))$

$$\partial_t\big(a\cdot u\big) + \partial_x\big(a\cdot F(u)\big) = a\cdot\mathcal{S}(u).$$

Subtracting this identity from the entropy inequality $\partial_t\eta(u) + \partial_x q(u) \leq \nabla\eta(u)\cdot\mathcal{S}(u)$ yields

$$\partial_t\Big(\eta(u) - a\cdot u\Big) + \partial_x\Big(q(u) - a\cdot F(u)\Big) \leq \big(\nabla\eta(u) - a\big)\cdot\mathcal{S}(u).$$

Subtracting the constants $\eta(u^*) - a\cdot u^*$ and $q(u^*) - a\cdot F(u^*)$ (which disappear under ∂_t and ∂_x) gives the distributional inequality

$$\partial_t\eta(u|u^*) + \partial_x q(u|u^*) \leq \big(\nabla\eta(u) - \nabla\eta(u^*)\big)\cdot\mathcal{S}(u).$$

In the particular case $\mathcal{S}_2(u) = -gH(kV/H^\alpha - S_b)$ and for the steady state satisfying $kV^*/H^{*\alpha} - S_b = 0$, one has $\partial_Q\eta(u) = V$ and $\partial_Q\eta(u^*) = V^*$, hence

$$\big(\nabla\eta(u) - \nabla\eta(u^*)\big)\cdot\mathcal{S}(u) = -(V-V^*)gH\big((kV/H^\alpha - S_b)\big) = -gk\,H(V/H^\alpha - V^*/H^{*\alpha})(V-V^*).$$

$$\partial_t\eta(u|u^*) + \partial_x q(u|u^*) \leq -gkH\left(\frac{V}{H^\alpha} - \frac{V^*}{(H^*)^\alpha}\right)(V - V^*)\,, \tag{2.37}$$

where

$$\begin{aligned} q(u|u^*) :&= q(u) - q(u^*) - \nabla\eta(u^*)\cdot\big(F(u) - F(u^*)\big) \\ &= \frac{1}{2}\left(g(H-H^*)\big(2Q - V^*(H+H^*)\big) + \frac{Q}{H^2}\,(Q - HV^*)^2\right). \end{aligned} \tag{2.38}$$

Since there exists a space of zero-measure $\mathcal{N}$ such that, for any $t \in [0,T]\setminus\mathcal{N}$, $u(t) \in BV(0,L)$ is defined and t is a Lebesgue point of $t \to W(u(t,\cdot))$. Let $\tau \in [0,T]\setminus\mathcal{N}$ and define $\psi_\varepsilon \in W^{1,\infty}(0,T)$ by

$$\psi_\varepsilon(t) = \begin{cases} 1, & 0 \leq t \leq \tau, \\ \dfrac{\tau+\varepsilon-t}{\varepsilon}, & \tau < t < \tau+\varepsilon, \\ 0, & t \geq \tau+\varepsilon, \end{cases}$$

so that $\psi_\varepsilon \geq 0$, $\psi_\varepsilon(0) = 1$, $\psi_\varepsilon(T) = 0$ and $\psi'_\varepsilon = -1/\varepsilon$ on $(\tau, \tau+\varepsilon)$. We now apply (2.37) to a sequence of non-negative $\psi_{\varepsilon,\delta} \in C^1_c([0,T); C^1([0,L]))$ such that for any $t \in [0,T)$

$$\lim_{\delta\to 0} \|\psi_{\varepsilon,\delta}(t,\cdot) - \psi_\varepsilon(t)\|_{C^1([0,L])} = 0 \tag{2.39}$$

so as to obtain at the limit

$$\begin{aligned} \iint_{(0,T)\times(0,L)} \eta(u|u^*)\, \psi'_\varepsilon(t)\, dx\, dt - gk \int_0^T \int_0^L H\left(\frac{V(t)}{H^\alpha} - \frac{V^*}{(H^*)^\alpha}\right)(V - V^*)\, dx\, \psi_\varepsilon(t)\, dt \\ + \int_0^T \Big(q(u(t,0+)|u^*) - q(u(t,L-)|u^*)\Big)\, \psi_\varepsilon(t)\, dt + \int_0^L \eta(u(0,\cdot)|u^*)\, dx \;\geq\; 0. \end{aligned} \tag{2.40}$$

Using that $\psi'_\varepsilon(t) = -1/\varepsilon$ on $(\tau, \tau+\varepsilon)$ and $\psi'_\varepsilon(t) = 0$ elsewhere, we compute

$$\iint \eta(u|u^*)\, \psi'_\varepsilon(t)\, dx\, dt = -\frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} \int_0^L \eta(u(t,x)|u^*)\, dx\, dt = -\frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} W(u(t,\cdot))\, dt.$$

Therefore (2.40) becomes

$$\begin{aligned} \frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} W(u(t,\cdot))\, dt + gk \int_0^T \int_0^L H\left(\frac{V(t)}{H^\alpha} - \frac{V^*}{(H^*)^\alpha}\right)(V - V^*)\, dx\, \psi_\varepsilon(t)\, dt \\ \leq W(u(0,\cdot)) + \int_0^T \Big(q(u(t,0+)|u^*) - q(u(t,L-)|u^*)\Big)\, \psi_\varepsilon(t)\, dt. \end{aligned} \tag{2.41}$$

Finally, letting $\varepsilon \to 0^+$ and using that

- τ is a Lebesgue point of $t \mapsto W(u(t,\cdot))$, hence
$$\frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} W(u(t,\cdot))\, dt \longrightarrow W(u(\tau,\cdot));$$
- $0 \leq \psi_\varepsilon \leq 1$ and $\psi_\varepsilon \to \mathbf{1}_{[0,\tau]}$ pointwise,

we have

$$\begin{aligned} W(u(\tau,\cdot)) + gk \int_0^\tau \int_0^L H\left(\frac{V(t)}{H^\alpha(t)} - \frac{V^*}{(H^*)^\alpha}\right)(V(t) - V^*)\, dxdt \\ \leq W(u(0,\cdot)) + \int_0^\tau \left(q(u(t,0+)|u^*) - q(u(t,L-)|u^*)\right) dt. \end{aligned} \tag{2.42}$$

Observe that, from our assumption (2.21)

$$(q(u(t,0+)|u^*) \leq 0 \;\text{ and }\; q(u(t,L-)|u^*)) \geq 0. \tag{2.43}$$

As a consequence

$$W(u(\tau,\cdot)) + gk \int_0^\tau \int_0^L H\left(\frac{V(t)}{H^\alpha} - \frac{V^*}{(H^*)^\alpha}\right)(V - V^*)\, dxdt \leq W(u(0,\cdot)). \tag{2.44}$$

Substituting by the expression (2.33) gives directly (2.22). This ends the proof of Proposition 2.1. □

Proof of Proposition 2.2. Let $u \in L^\infty((0,\infty); BV(0,L))$ be an entropic solution of (2.1), (2.2) with initial condition u^0. It follows from Helly's theorem (see [19, Theorem 2.3]) that for any a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $t_k \to +\infty$ one can extract a subsequence (that we denote again by t_k) such that there exists $\bar{u} \in BV(0,L)$ and

$$u(t_k,\cdot) \to \bar{u} \text{ a.e. as } k \to +\infty, \tag{2.45}$$

and therefore $u(t_k,\cdot) \to \bar{u}$ in $L^1(0,L)$. Thus, the trajectories of the entropic solution of (2.1), (2.2) are precompact in $L^1(0,L)$ (because the set of trajectories is relatively compact in L^1, note that this can hold only because $u \in L^\infty((0,\infty); BV(0,L)))$.

Let us denote by ω_0 the space of invariant trajectories, that is

$$\omega_0 := \{u = (H,V) \in L^\infty((0,T); BV(0,L)) \mid u \text{ is an entropic solution of (2.1), (2.2)} \\ \text{and } W(u(t,\cdot)) = W(u(0,\cdot)),\ \forall \text{ a.e. } t \in [0,T]\}. \tag{2.46}$$

Now we assume that $u \in \omega_0$ with $H(0,x) > 0$ a.e.. By the estimate (2.44) with $\alpha = 0$, we have that

$$\|H(t,\cdot)(V(t,\cdot) - V^*)^2\|_{L^1} = 0. \tag{2.47}$$

As a consequence,

$$H(V - V^*) = 0, \quad \text{a.e. on } (0,T) \times (0,L). \tag{2.48}$$

Hence $Q = HV = V^*H$, a.e. on $(0,T) \times (0,L)$. In addition, this means that

$$gH(kV(t) - S_b) = gkH(V(t) - V^*) = 0 \text{ a.e. on } (0,T) \times (0,L). \tag{2.49}$$

Since (H,Q) is a weak solution of (2.31), and using (2.49), one has in a distributional sense

$$\begin{aligned} &\partial_t H + V^* \partial_x H = 0, \\ &V^* \left(\partial_t H + V^* \partial_x H\right) + g\partial_x \left(\frac{H^2}{2}\right) = 0. \end{aligned} \tag{2.50}$$

and therefore, using the first equation in the second one, $\partial_x(H^2) = 0$ which means that H^2 is a constant with respect to x and from the first equation it is also constant with respect to time. Thus there exists $\bar{H}$ such that $H = \bar{H}$ a.e. (and as a consequence everywhere) on $(0,T) \times (0,L)$. From the boundary conditions, this implies in particular that

$$G_0(\bar{H}) = G_1(\bar{H}) = \frac{S_b \bar{H}}{k}, \tag{2.51}$$

which, from (2.29), implies that $\bar{H} = H^*$.

Thus, $\omega_0 = \{(H^*, V^*)\}$. Consequently, (2.30) follows from Dafermos invariance principle (see [30]), also referred to as an the infinite-dimensional version of LaSalle's invariance principle [83, Theorem 3.1] or [81, Proposition 4.2, Remark 4.3]. This concludes the proof of Proposition 2.2.

□

This raises several questions that are still open:

- Can one show the (semi)-global well-posedness of this system? In other words, can one weaken the assumptions of Proposition 2.2 by showing the existence of an entropic solution in $C^0([0,+\infty); L^1(0,L)) \cap L^\infty([0,+\infty); BV(0,L))$?
- Can one get a global asymptotic stability / stabilization result in L^2 norm (with potentially regular but not small initial condition)?
- Is it possible to obtain a global stabilization in H^1? In this case, the solutions are more regular and some of the difficulty vanish, but one needs to find a good entropy.
- Is it possible to obtain a local stabilization in H^1 ? This is a less ambitious question than the previous one, but it is already open: so far the stabilization results only exists for stronger norms: H^p for $p \geq 2$ [12, 63] and C^q for $q \geq 1$ [56]. In fact, in general for hyperbolic systems the local stabilization by boundary controls likely cannot hold in H^1 norm so the particular structure is important.
- Is it possible to achieve a similar result for another steady-state than the constant steady-state?

3 Perspectives

In this section, we present several perspectives on the stabilization of Saint-Venant equations by boundary controls.

3.1 Proportional-Integral control

Proportional-Integral (PI) control have been widely used in practice to regulate and stabilize systems modelled by the Saint-Venant equations. In this situation, the boundary conditions (2.2) are replaced by

$$\begin{aligned} &H(L-L_i)V(L-L_i) = Q^*, \\ &H(L_i)V(L_i) = Q^* + k_p(H(L_i) - H^*(L_i)) - k_I Z, \end{aligned} \tag{3.1}$$

where Q^* is a given and imposed flow, potentially unknown, H_c is a target height, $\dot{Z} = (H(L_i) - H^*(L_i))$ and L_i can be either L or 0. The interest of this control is that it is robust to a constant error in the control, thanks to the integral part. This is called *constant disturbances rejection*. This very useful practical property comes at a price from a mathematical point of view: its analysis is typically more (and sometimes much more) challenging than the control (1.2), a difficulty that arise particularly for nonlinear infinite-dimensional systems (one can look at the length of the proof in [86, 26] even only for a transport equation) In the framework of Saint-Venant equations, this control was studied for instance in [88, 76, 77, 38, 40, 89, 85, 13, 58, 59] (see also [84, 87, 71, 72] for its use in other systems of partial differential equations).

In practice, and in the literature, the most common case, is $L_i = L$ in (3.1). This corresponds to an imposed incoming flow Q^* at $x = 0$ and a PI control at $x = L$. In this situation, a first result taking into account a potentially non-small source term S_f was shown by Bastin and Coron in [13]

Theorem 3.1 ([13]) *Assume that $S_b = 0$. Let (H^*, V^*) be a steady-state of* (2.1), (3.1) *satisfying* (1.4), *if*

$$k_p > 1, \ k_I < 0, \tag{3.2}$$

then (H^*, V^*) is exponentially stable for the H^2 norm.

Remark 3.1 We see another interesting feature of the PI control for the stabilization of the Saint-Venant equation: the sufficient conditions on the control parameters k_p and k_I do not depend on H^*, V^* (or equivalently on Q^* and H_c given that the value of Q^* and H_c impose the steady-state (H^*, V^*), see for instance [58]).

These conditions were later improved by

Theorem 3.2 ([58]) *Let (H^*, V^*) be a steady-state of* (2.1), (3.1) *satisfying* (1.4), *if*

$$\begin{aligned} & k_p > 0, \ k_I < 0, \\ or \quad & k_p < -\frac{gH^*(L) - V^{*2}(L)}{V^*(L)} \quad and \ \ k_I > 0 \end{aligned} \tag{3.3}$$

then (H^*, V^*) is exponentially stable for the H^2 norm.

The case $L_i = 0$ in (3.1) is more challenging: the best known results either deal with the homogeneous case or cases with a small source term. Nevertheless a necessary and sufficient condition was found by Bastin, Coron and Tamasoiu for the linearized system around a steady-state (H^*, V^*) satisfying (1.4) when $S_f = S_b = 0$

$$\begin{aligned} & \partial_t h + H^* \partial_x v + V^* \partial_x h = 0, \\ & \partial_t v + V^* \partial_x v + g \partial_x h = 0, \end{aligned} \tag{3.4}$$

The necessary and sufficient condition is given in the following theorem

Theorem 3.3 ([17]) *The system* (3.4) *is exponentially stable if and only if*

$$k_p \in \left(-\frac{(gH^* - V^{*2})}{V^*}, 0 \right) \quad and \ \ k_I \in \left(0, \omega_0 \sin(\omega_0 \tau) \frac{(V^* - k_p)(gH^* - V^{*2})}{2\sqrt{gH^*} V^*} \right), \tag{3.5}$$

where ω_0 is the smallest positive ω such that

$$\cos(\omega \tau) = \frac{(\sqrt{gH^*} - V^*)^2 (\sqrt{gH^*} + V^* - k_p) + (\sqrt{gH^*} + V^*)^2 (V^* - \sqrt{gH^*} - k_p)}{(gH^* - V^{*2})(2k_p - 2V^*)}. \tag{3.6}$$

Open questions

- For the nonlinear system (2.1), (3.1) with $L_i = 0$ (the control is at $x = 0$) and in the case $S_f = S_b = 0$ it is possible to show that under the conditions (3.5), (3.6) the system is exponentially stable for the H^2 norm (or another suitable norm)? Since the exponentially stability of the linearized system is known this is a natural question but, interestingly, it could be a challenging question and it is not completely clear that the answer is yes. For a scalar system an extraction method was used to recover the stability of the nonlinear system from the known conditions on the stability of the linearized system in [26] but this method requires lengthy computations. The question of its application to the Saint-Venant equations is still open.

- When S_f and S_b are generic and not necessarily identically zero and $L_i = L$, are the conditions (3.3) necessary for the stability of the system (2.1), (3.1) or can one find even less restrictive boundary conditions ?
- An alternative, maybe more approachable: are the conditions (3.3) necessary for the existence of a basic Lyapunov function for the H^2 norm?
- What can be done in the C^1 norm (C^q norms for $q \geq 1$)?

3.2 Boundary control of the linearized Saint-Venant-Exner equations

The Saint-Venant–Exner (SVE) equations describe the coupled dynamics of shallow water flow and sediment transport (bathymetry evolution) in open channels. The nonlinear model is typically written as a system of balance laws [68]:

$$\begin{aligned}
&\partial_t H + V\partial_x H + H\partial_x V = 0,\\
&\partial_t V + V\partial_x V + g\partial_x H + g\partial_x B - gS_b + C_f\frac{V^2}{H} = 0,\\
&\partial_t B + aV^2\partial_x V = 0,
\end{aligned} \tag{3.7}$$

where H, V, and B denote the water depth, water velocity, and bathymetry (sediment layer depth), respectively. The parameters g, S_b, C_f, and a represent the gravity constant, channel bottom slope, friction coefficient, and a parameter encompassing porosity and viscosity effects on sediment dynamics.

One can observe that for a smooth steady-state with nonzero velocity V^* and H^* are constants and

$$\partial_x B = gS_b - C_f\frac{V^2}{H}. \tag{3.8}$$

In particular, this partially addresses the important Saint-Venant equations' challenge of inhomogeneous steady states, but introduces a new complexity: an additional equation. Because the sedimentation is typically a slow phenomenon compared to the water flow, a is often assumed to be small. Under this assumption, the propagation speeds of this system can be approximated for small a as

$$\lambda_1 \approx V^* + \sqrt{gH^*}, \quad \lambda_2 \approx \frac{agV^{*3}}{gH^* - V^{*2}}, \quad \lambda_3 \approx V^* - \sqrt{gH^*}. \tag{3.9}$$

A defining feature of this system is its *heterodirectional* nature. For a subcritical flow regime ($gH^* > V^{*2}$), the eigenvalues satisfy $\lambda_3 < 0 < \lambda_2 \ll \lambda_1$, where λ_1 and λ_3 represent the faster hydrodynamic waves, and λ_2 represents the much slower sediment transport wave. For a supercritical flow ($gH^* < V^{*2}$), the signs change (e.g., $\lambda_2 < 0 < \lambda_3 < \lambda_1$), but the bidirectional propagation remains.

In [11, 33], the authors study the boundary control problem for the linearized Saint-Venant-Exner equations with constant steady-state (i.e. assuming additionally that B is constant) via hydraulic devices (e.g., gates or pumps) and is formulated by relating the incoming characteristics to the outgoing ones via a feedback gain matrix K. More precisely, the linearized system around the constant steady-state (V^*, H^*, B^*) is reformulated as

$$\partial_t \xi + \Lambda\partial_x \xi + M\xi = 0, \quad x \in (0, L),\ t \geq 0, \tag{3.10}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ is the diagonal matrix of characteristic velocities, and the boundary control feedback is given by

$$\xi_{in}(t) = K\xi_{out}(t), \tag{3.11}$$

where ξ_{in} and ξ_{out} denote the vectors of incoming and outgoing characteristics at the boundaries $x = 0$ and $x = L$, respectively. The result is the following:

Theorem 3.4 ([11, 33]) *Assume that there exists a diagonal positive definite matrix P such that $M^\top P + PM \geq 0$ and $\|\Delta K \Delta^{-1}\| < 1$ with $\Delta := (P|\Lambda|)^{\frac{1}{2}}$, then the equilibrium $\xi \equiv 0$ of the system* (3.10), (3.11) *is exponentially stable in the L^2-norm.*

Recently, [80] revisited this problem using a fundamentally different mathematical framework based on operator semigroup theory. They proved that the well-posedness and exponential stability of (3.10), (3.11) can be established by verifying the dissipativity of the system operator and its adjoint (via the Lumer–Phillips theorem). To overcome the restrictive assumption on the source term matrix M, [34] used the PDE backstepping method, where the control is a feedback of the full-state or observer. Instead of searching for a Lyapunov function, this approach constructs a Volterra integral transformation that maps the original SVE system into a strictly stable target system.

Open questions

- Is it always possible to stabilize the Saint-Venant-Exner equations with a boundary control K, whatever S_b and C_f and the length of the domain L? While this property holds for the Saint-Venant equation (see [62, 63]), it is not known for the Saint-Venant-Exner equation. Otherwise, is it possible to give a explicit sufficient condition involving the source term M and the boundary term K for the existence of a Lyapunov function and what are the corresponding limits on S_b and C_f ?
- Can one show the stabilization of the nonlinear system in the H^2 norm or in C^1 norm?
- Is it possible to consider the stabilization of the entropy solution of the balance law (3.7)?

3.3 Saint-Venant equations with viscosity

Saint-Venant equations are traditionally derived from Navier-Stokes equations under a shallow-water approximation (see for instance [41, Proposition 4.1]). Interestingly, the viscosity disappear at the first order approximation, leading to hyperbolic equations such as (1.1). However, when looking at a higher order approximation, the viscosity manifest itself and we end up with the so-called *viscous Saint-Venant equations*. Such equations were derived in [41] and are written

$$\begin{aligned} &\partial_t H + \partial_x(HV) = 0, \\ &\partial_t V + V\partial_x V + g\partial_x H + \frac{f(H,V)}{H} - 4\mu\frac{\partial_x(H\partial_x V)}{H} = 0. \end{aligned} \tag{3.12}$$

In this situation one need an additional boundary condition for the system to be well-posed, because of the second-order derivative term in the second equation (see [61]), for instance

$$\begin{aligned} H(t,0)V(t,0) &= \mathcal{G}_1(H(t,0)), \\ H(t,L)V(t,L) &= \mathcal{G}_2(H(t,L),\partial_x V(t,L)), \\ \partial_x V(t,0) &= 0. \end{aligned} \tag{3.13}$$

One can naturally wonder if, provided that the viscosity μ is small, one can recover the results of Theorem 1.1 and/or Section 2. In turns out that the answer is –at least partially– no. Interestingly the existence of a function η solution to (2.6) is not anymore sufficient for the existence of a basic Lyapunov function. Even in the linearized case, most functions $\eta > 0$ satisfying (2.8) do not allow to create a Lyapunov function[6] and, in particular, the Lyapunov function found in [62] and [60] do not work anymore.

Nevertheless, the Lyapunov function of [12], derived from the mechanical energy, can still in some cases lead to the exponential stability of the linearized system given formally by

$$\begin{aligned} &\partial_t h + V^*\partial_x h + H^*\partial_x v + (\partial_x V^*)h + (\partial_x H^*)v = 0, \\ &\partial_t v + \left(g - 4\mu\frac{\partial_x V^*}{H^*}\right)\partial_x h + \left(V^* - 4\mu\frac{\partial_x H^*}{H^*}\right)\partial_x v - 4\mu\partial_{xx} v \\ &\qquad + \frac{f_H(H^*,V^*)H^* - f(H^*,V^*) + 4\mu\partial_x H^*\partial_x V^*}{H^{*2}}h + \left(\partial_x V^* + \frac{f_V(H^*,V^*)}{H^*}\right)v = 0, \\ &v(t,0) = b_0 h(t,0), \\ &v(t,L) = b_1 h(t,L) + \mu c_1 \partial_x v(t,L), \\ &\partial_x v(t,0) = 0, \end{aligned} \tag{3.14}$$

where

$$b_0 = \frac{\mathcal{G}_1'(H^*(0)) - V^*(0)}{H^*(0)},\ b_1 = \frac{\partial_1\mathcal{G}_2(H^*(L),\partial_x V^*(L)) - V^*(L)}{H^*(L)},\ c_1 = \frac{\partial_2\mathcal{G}_2(H^*(L),\partial_x V^*(L))}{H^*(L)}. \tag{3.15}$$

This is illustrated by the following theorem

Theorem 3.5 ([61]) *Let (H^*,V^*) be a steady state of* (3.12), (3.13) *and satisfying* (1.4)*. If, in addition,*

$$gH^*(0) < (2+\sqrt{2})V^{*2}(0), \tag{3.16}$$

then if

$$b_0 \in (b_0^-, b_0^+), \quad b_1 \in \mathbb{R}\setminus[b_1^-, b_1^+], \quad c_1 \in (c_1^-, c_1^+), \tag{3.17}$$

where

$$b_0^{\pm} = g\left(-\frac{1}{V^*(0)} \pm \sqrt{\frac{1}{V^*(0)^2} - \frac{1}{gH^*(0)}}\right), \quad b_1^{\pm} = g\left(-\frac{1}{V^*(L)} \pm \sqrt{\frac{1}{V^*(L)^2} - \frac{1}{gH^*(L)}}\right),$$

$$c_1^{\pm} = \frac{4\left(-V^*(L) - b_1 H^*(L) \pm \sqrt{V^*(L)(H^*(L)V^*(L)b_1^2/g + 2H^*(L)b_1 + V^*(L))}\right)}{gH^*(L) - V^*(L)^2}.$$

the linearized system (3.14) *is exponentially stable (for the L^2 norm).*

[6] this is a consequence of the correspondance between η and the Lyapunov function given in [10] and [61, Proposition 3.1]

Other stabilization results using boundary conditions exist such as [2] which uses moving walls as a boundary control. While it is known that an arbitrarily small addition of viscosity can completely change the behavior of such a 2×2 hyperbolic system (see for instance [15, 16]), this raises several open questions:

- Can one show the stabilization of the nonlinear system in the H^2 norm? This is expected given the existence of a basic Lyapunov function, but it requires a proof of the well-posedness in the nonlinear setting.
- Can one show similar results where there is a slope? Can one always find sufficient stability conditions on G_0, G_1 to stabilize the system whatever the length of the domain $L > 0$, as in the inviscid case?
- What can be done in the C^1 (or C^q) norm? There are preliminary results from [38] using a characteristic approach but the problem is still very much open.

3.4 Networks of Saint-Venant equations

Many flow systems modeled by Saint-Venant equations are actually composed of a network of branches, each of which is modeled by an instance of Saint-Venant equations. This is the case, for example, with waterways and irrigation networks. For this reason, considering networks of Saint-Venant systems has been considered for decades (see for instance [88, 69, 73, 74, 18, 89, 38], see also [46, 47, 79, 37, 45, 49]). Several of the results described above generalize to systems of Saint-Venant networks. For instance, consider the following networks of N Saint-Venant systems

$$\begin{aligned} &\partial_t A_i + \partial_x (A_i V_i) = 0, \\ &\partial_t (A_i V_i) + \partial_x (A_i V_i^2) + gA(\partial_x H_i - S_b(x) + S_f(A_i, V_i, x)) = 0, \end{aligned} \tag{3.18}$$

with the boundary conditions

$$\begin{aligned} A_1(t,0)V_1(t,0) = Q^*, \ \ A_i(t,0)V_i(t,0) &= A_{i-1}(t,L_i)V_i(t,L_{i-1}) \\ A_i(t,L_i)V_i(t,L_i) &= Q^* + k_{p,i}(A_i(t,L_i) - A_i^*(L_i)) - k_{I,i}Z_i, \\ \dot{Z}_i(t) &= A_i(t,L_i) - A_i^*(L_i), \end{aligned} \tag{3.19}$$

where the first equation holds for any $i \in \{2, ..., N\}$ and the two following equations holds for any $i \in \{1, ..., N\}$. These boundary conditions represent the simplest type of network: *a network in cascade* where the branches are one after the other and at each junction there is exactly one entering branch and one out going branch. In this framework, one has the following result

Theorem 3.6 ([59]) *Let $\{(H_i^*, V_i^*)\}_{i\in\{1,...,N\}}$ be a steady state of* (3.18), (3.19) *satisfying* (1.4) *with $(H_i^*,\ V_i^*)$ instead of (H^*, V^*) for any $i \in \{1, ..., N\}$. If for any $i \in \{1, ..., N\}$ the parameters $k_{p,i}$, $k_{I,i}$ satisfy* (3.3) *with $(H_i^*, V_i^* k_{p,i}, k_{I,i})$ instead of (H^*, V^*, k_p, k_I), then the steady-state $\{(H_i^*, V_i^*)\}_{i\in\{1,...,N\}}$ is exponentially stable for the H^2 norm.*

This is only an example among many others, permitted by the fact that a cascade network is a relatively simple extension of the single branch case. However, in practice, many networks have more complex geometry which, in turn, are more challenging. When the network has a *tree structure*[7], several results are known [43, 44, 60].

In particular, one can represent a tree-shape network as follows:

- Let the N channels be indexed by $i \in \{1, \dots, N\}$.
- Denote by A the root node, that is the starting point of the network and label the trunk channel as $i = 1$.
- Denote by J_M^i the multiple nodes where i is the index of the incoming channel and by $\mathcal{I}_{out}^i$ the set of indices corresponding the outgoing channels as this multiple node.
- Denote by J_S^i the simple nodes at the ends of the network where i is the index of the incoming channel.
- Denote by $\mathcal{M}$ the set of indices of channels that end at a multiple node
- Denote by $\mathcal{S}$ the set of channels that end at a simple node.

A tree-shape network can be represented as systems (3.18) with $S_P = \mathrm{Id}$, $S_f = \frac{C_i V_i^2}{H_i}$, $S_b = 0$, with the boundary conditions

$$\begin{aligned}
A &: H_1(t,0)V_1(t,0) = Q_1, \\
J_M^i &: H_i(t,L_i)V_i(t,L_i) = \sum_{j \in \mathcal{I}_{out}^i} H_j(t,0)V_j(t,0), \quad i \in \mathcal{M}, \\
&\quad H_i(t,L_i) = H_j(t,0), \quad i \in \mathcal{M}, \quad j \in \mathcal{I}_{out}^i, \\
J_S^i &: H_i(t,L_i)V_i(t,L_i) = \mathcal{G}_i(H_i(t,L_i)), \quad i \in \mathcal{S}.
\end{aligned} \tag{3.20}$$

Theorem 3.7 ([60]) *Let $\{(H_i^*, V_i^*)\}_{i \in \{1,\dots,N\}}$ be a steady state of* (3.18), (3.20) *satisfying* (1.4) *with $(H_i^*,\ V_i^*)$ instead of (H^*, V^*) for any $i \in \{1, ..., N\}$. For any $i \in \{1, ..., N\}$, let*

$$b_{1i} = \frac{\mathcal{G}_i'(H_i^*(L_i))H_i^*(L_i) - \mathcal{G}_i(H_i^*(L_i))}{H_i^{*2}(L_i)}. \tag{3.21}$$

If, for any $i \in \{1, ..., N\}$, the boundary controls $\mathcal{G}_i$ ($i \in \mathcal{S}$) satisfy (2.11) *with $(H_i^*, V_i^*, b_{1i}, \lambda_{1i}, \lambda_{2i}, m_i)$ instead of $(H^*, V^*, b_1, \lambda_1, \lambda_2, m)$, then the steady-state $\{(H_i^*, V_i^*)\}_{i \in \{1,\dots,N\}}$ is exponentially stable for the H^2 norm.*

Remark 3.2 Remarkably, this result shows that a tree-shape network can be stabilized only with controls at the leaves of the tree, i.e. the outflows extremities of the network.

Other networks, even more challenging, have been considered: several studies for instance where devoted to the controllability or the nodal controllability of networks that may have cycles (and hence not in a tree structure), see for instance [75, 91, 45] and the references therein. There are few stabilization results for the Saint-Venant system in this case, in particular when the source term is not identically zero, and one can cite the result of [45] that underlines that there are intrinsic limitations when there is no control at internal nodes.

[7] meaning that a channel can divides itself in several but two channel cannot merge in one, or in other words there is only one channel entering at each node but potentially several channel that exit the node

3.5 Multi-dimensional Saint-Venant equations

Because Saint-Venant equations are derived from Navier-Stokes equation under a shallow water approximation, they exist both in one dimension of space (coming from 2D Navier-Stokes) and two dimension in space (coming from 3D Navier-Stokes). In their version in two dimension of space, the Saint-Venant equations are used to model, for instance, the atmosphere and can be written

$$
\begin{aligned}
&\partial_t H + \partial_x(HU) + \partial_y(HV) = 0,\\
&\partial_t U + U\partial_x U + V\partial_y U + g(\partial_x H + S_f^U(H,U,V,x,y) - \partial_x S_b(x,y)) - lV = 0,\\
&\partial_t V + U\partial_x V + V\partial_y V + g(\partial_y H + S_f^V(H,U,V,x,y) - \partial_y S_b(x,y)) + lU = 0,
\end{aligned}
\tag{3.22}
$$

where H is the water depth, U is the velocity in the x-direction, V is the velocity in the y-direction, S_f^U, S_f^V are the source terms taking into account the friction, $\partial_x S_b, \partial_y S_b$ are the bottom slopes of the channel in the x-direction and in the y-direction, and l is the Coriolis coefficient associated with the Coriolis force.

Most of the methods designed to stabilize quasilinear hyperbolic systems with boundary controls are very specific to the dimension one in space. From the basic Lyapunov function to the (Volterra) backstepping, the existing methods often heavily rely on the fact that the transport term can be diagonalized without loss of generality in dimension one in space. Roughly speaking, this means that when one has a system of the form

$$
\partial_t w + A\partial_x w + Mw = 0,
\tag{3.23}
$$

A is diagonalizable for the system to be hyperbolic and thus A can be assumed diagonal without loss of generality. This does not hold anymore in space dimension 2 and above: indeed if one has instead

$$
\partial_t w + A\partial_x w + B\partial_y w + Mw = 0 \quad \text{in } [0,+\infty)\times\Omega,
\tag{3.24}
$$

with symmetric A, B, the hyperbolicity condition imposes that $\mathcal{A}(\nu) = A\nu_1 + B\nu_2$ is diagonalizable for any $\nu = (\nu_1,\nu_2) \in \mathbb{S}$ but this does not imply that A and B have to be co-diagonalizable and, in particular, we cannot assume without loss of generality that they are both diagonal up to a change of variables. As a consequence, multidimensional systems are a major challenge[8], which makes them all the more interesting.

Some approaches have been developed in the very recent years [32, 64, 65, 90][9] and Saint-Venant equations have been used as a natural testbed. In contrast to the 1D case, the results are very scarce and both [65] and [90] provide a Lyapunov function for the L^2 norm with uniform weight under dissipative structure assumption for (3.24). More specifically, the result of [65] is the following[10].

[8] except in the particular cases where the transport term can be diagonalized again, that is A and B are co-diagonalizable in (3.24)

[9] Another interesting aspect of this section is that it is the only one where there is no contribution from Jean-Michel Coron

[10] The results in [65] are in fact established for multi-dimensional systems with coefficients that may depend on the spatial variables. Here, we only present the version for two-dimensional, constant-coefficient systems.

Theorem 3.8 ([65]) *Let $w(t,x,y) \in C^1([0,T], H^s(\Omega^2))$, $s \geq 2$, be a solution to the system* (3.24)*. Assume there exist $m_1, m_2 \in \mathbb{R}$ such that*

$$C\mathrm{Id} + m_1 A + m_2 B - M - M^\top \leq 0, \quad C > 0. \tag{3.25}$$

Define the Lyapunov function for system (3.24) *as follows*

$$L(w(t)) = \int_\Omega w(t,x,y)^\top w(t,x,y) e^{m_1 x + m_2 y} dxdy. \tag{3.26}$$

Assume that the boundary conditions are such that[11]

$$\int_{\partial\Omega} \exp(m_1 x + m_2 y) w^T(t,x,y) \left(n_1(x,y)A + n_2(x,y)B\right) w(t,x,y) dxdy \geq 0, \tag{3.27}$$

where (n_1, n_2) is the outward unit normal vector to $\partial\Omega$. Then we have

$$\frac{d}{dt} L(t) \leq -CL(t). \tag{3.28}$$

Further, the solution w of the system (3.24) *with initial and boundary conditions is exponentially stable in the L^2-sense.*

A detailed description of the boundary conditions can be found in [65].

In [90], the stability is proved under the structural stability condition:

$$M = \begin{pmatrix} 0 & 0 \\ 0 & M_0 \end{pmatrix} \quad \text{with } M_0 \in \mathbb{R}^{r \times r} \text{ satisfying } M_0 + M_0^\top > 0. \tag{3.29}$$

Let

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2^\top & A_0 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_2 \\ B_2^\top & B_0 \end{pmatrix}$$

have the same block structure as M. In this framework, one has the following result

Theorem 3.9 ([90]) *Assume that* (3.29) *holds and that there exist real numbers m_1, m_2 such that*

$$m_1 A_1 + m_2 B_1 < 0. \tag{3.30}$$

Then there exist proper boundary conditions, i.e. boundary conditions ensuring that

$$\int_{\partial\Omega} (K + m_1 x + m_2 y) w(t,x,y)^T A_0 \left(n_1(x,y)A + n_2(x,y)B\right) w(t,x,y) dxdy \geq 0, \tag{3.31}$$

for some $K > m_1 + m_2$, such that the corresponding solutions w to system (3.24) *with initial conditions are exponentially stable in L^2-sense.*

The Lyapunov function given by [90] is

$$L(w(t)) = \int_\Omega w(t,x,y)^\top w(t,x,y)(K + m_1 x + m_2 y) dxdy, \tag{3.32}$$

with positive constant K chosen in the proof.

In particular, they show the following

[11] Contrarily to the 1D-case, the existence of proportional boundary conditions satisfying this condition is not granted, this motivates this assumption.

Corollary 3.1 ([90]) *Assume that* $S_f^U = kU$, $S_f^V = -kV$, $L_y = 1$ *and denote* $L_x = L$. *Consider a constant steady-state* (H^*, U^*, V^*) *with* $V^* = 0$ *and the linearized Saint-Venant equations around the steady-state* $(H^*, U^*, 0)$

$$\begin{aligned} &\partial_t h + H^*(\partial_x u + \partial_y v) + U^* \partial_x h = 0, \\ &\partial_t u + U^* \partial_x u + g\partial_x h + gku - lv = 0, \\ &\partial_t v + U^* \partial_x v + g\partial_y h + gkv + lu = 0, \end{aligned} \tag{3.33}$$

If the boundary conditions are such that the system is well-posed in $C^0([0,T]; L^2((0,L)\times(0,1)))$ *and for any solution* (h,u,v) *one has*

$$\begin{aligned} &-2g\int_0^L [(2L-x)(hv)]_{y=0}\,dx + 2g\int_0^L [(2L-x)(hv)]_{y=1}\,dx \\ &+ LU^*\int_0^1 \left[v^2 + \frac{g}{H^*}h^2 + 2\frac{g}{U^*}hw + w^2\right]_{x=L} dy \\ &- 2LU^*\int_0^1 \left[v^2 + \frac{g}{H^*}h^2 + 2\frac{g}{U^*}hw + w^2\right]_{x=0} dy \geq 0 \end{aligned} \tag{3.34}$$

then the linearized Saint-Venant equations (3.33) *with these boundary conditions are exponentially stable in the* L^2*-sense.*

One can notice that both works require dissipativity assumptions, with [65] assuming (3.25) and [90] assuming (3.29)–(3.30). One can note that (3.29)–(3.30) actually implies (3.25). Both Lyapunov functions (3.26) and (3.32) have uniform weight. As expected, dissipativity assumptions are rather restrictive. In particular, one can find that the following basic example does not satisfy any of these dissipativity assumptions

$$\partial_t w + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x w + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_y w = 0. \tag{3.35}$$

Recent work [6] shows that this limitation can be overcome by considering Lyapunov functions for higher norm, such as H^1 norm. For two matrices M_1 and M_2, denote by $[M_1, M_2]$ the commutator, namely, $[M_1, M_2] = M_1M_2 - M_2M_1$. Consider the following Lyapunov candidate for system (3.24) with $M = 0$:

$$V(w) = V_{H^1}(w) + V_{L^2}(w), \tag{3.36}$$

where

$$V_{H^1}(w) = \int_\Omega \begin{pmatrix} w_x \\ w_y \end{pmatrix}^\top \begin{pmatrix} Q_A & Q_C^\top \\ Q_C & Q_B \end{pmatrix} \begin{pmatrix} w_x \\ w_y \end{pmatrix} dxdy, \quad V_{L^2}(w) = \int_\Omega w^\top Q w dxdy, \tag{3.37}$$

with $\begin{pmatrix} Q_A & Q_C^\top \\ Q_C & Q_B \end{pmatrix}$ and Q being positive definite. We have the following result:

Theorem 3.10 ([6]) *Let* $S_C = Q_C + Q_C^\top$. *The functional* V *defined in* (3.36) *is a Lyapunov function for the* H^1 *norm if*

(i) The commutativity conditions

$$[A, Q_A] = 0, \quad [B, Q_B] = 0, \quad [A, S_C] + [B, Q_A] = 0, \quad [B, S_C] + [A, Q_B] = 0, \tag{3.38}$$

hold;

(ii) The quadratic matrix function in the integral term of $\dot{V}$ is negative definite within Ω;

(iii) The quadratic matrix function in the boundary term of $\dot{V}$ is negative semi-definite on $\partial\Omega$.

Here we omit the (heavy) explicit expressions of conditions (*ii*) and (*iii*) in the Theorem 3.10 that can be found in [6, Theorem 3.1]. These two conditions are the analogous of the interior condition and the boundary conditions one would obtain in 1-D (see [11, Chapter 6]). One can show that the commutativity conditions (3.38) provide in general richer possibilities for the choice of Lyapunov function for the H^1 norm (see [6]).

While these Lyapunov functions are in a higher norm, in many cases they can be used to define a Lyapunov function for the L^2 norm (see [6] or [27]). In particular Theorem 3.10 allows to show the stability of the system (3.35) in the L^2 norm with boundary conditions of the form

$$\begin{cases} w_1 = k_1 w_2, & \text{on } \{x=0\}, \\ w_2 = k_2 w_1, & \text{on } \{x=1\}, \\ w_1 + w_2 = k_3(w_1 - w_2), & \text{on } \{y=0\}, \\ w_1 - w_2 = k_4(w_1 + w_2), & \text{on } \{y=1\}, \end{cases} \tag{3.39}$$

provided some sufficient conditions on the constants k_i (see [6, Definition 3.9]).

There many interesting open questions, among them:

- Can we find a proposition similar to Proposition 2.2 for the two dimensional system?
- Can the result of [6] be applied to the two dimensional Saint-Venant equations?
- Provided L_y is bounded and not too large (or even sufficiently small) can we stabilize the system by the boundary with proportional controls for any length of the domain L_x ?

4 Conclusion

In this paper, we have surveyed progress on the boundary stabilization of the Saint-Venant equations over the past three decades, a rich problem with many foundational contributions by Jean-Michel Coron. We have also discussed several new perspectives related to this topic. Although the problem has been studied for several decades, many important questions remain open and we summarized several of the main ones which, we believe, could offer promising avenues for future research.

A Necessary condition for the homogeneous case

Assume that (2.5) holds. Consider the Riemann coordinates $(w_1, w_2) = (V + 2\sqrt{gH}, V - 2\sqrt{gH})$. Then system (2.1) reads

$$\begin{aligned} \partial_t w_1 + \frac{1}{4}(3w_1 + w_2)\partial_x w_1 = 0, \\ \partial_t w_2 + \frac{1}{4}(3w_2 + w_1)\partial_x w_2 = 0, \end{aligned} \tag{A.1}$$

and the boundary conditions are

$$
\begin{aligned}
&\frac{(w_1(t,0)-w_2(t,0))^2}{16g}\frac{w_1(t,0)+w_2(t,0)}{2} = G_0\left(\frac{(w_1(t,0)-w_2(t,0))^2}{16g}\right),\\
&\frac{(w_1(t,L)-w_2(t,L))^2}{16g}\frac{w_1(t,L)+w_2(t,L)}{2} = G_1\left(\frac{(w_1(t,L)-w_2(t,L))^2}{16g}\right).
\end{aligned}
\tag{A.2}
$$

The boundary conditions (A.2) can be expressed locally as

$$
\begin{aligned}
w_1(t,0) &= w_1^*(0) + k_0(w_2(t,0) - w_2^*(t,0)) + O(|w(t,0)-w^*(0), w(t,L)-w^*(L)|),\\
w_2(t,L) &= w_2^*(L) + k_1(w_1(t,L) - w_1^*(L)) + O(|w(t,0)-w^*(0), w(t,L)-w^*(L)|),
\end{aligned}
\tag{A.3}
$$

where $(w_1^*, w_2^*) = (V^*+2\sqrt{gH^*}, V^*-2\sqrt{gH^*})$ and condition (2.5) implies that, there exists $\varepsilon > 0$ depending only on the steady-state (w_1^*, w_2^*) such that if $|w(t,0)-w^*(0), w(t,L)-w^*(L)| < \varepsilon$,

$$
|k_0 k_1| > 1,
\tag{A.4}
$$

Let

$$
\lambda_1(w) = \frac{1}{4}(3w_1 + w_2), \quad \lambda_2(w) = -\frac{1}{4}(3w_2 + w_1).
\tag{A.5}
$$

We have $\lambda_1(w^*) = \sqrt{gH^*} + V^* > 0$ and $\lambda_2(w^*) = \sqrt{gH^*} - V^* > 0$. Now consider the following candidate Lyapunov function

$$
V = V_1 + V_2 + V_3,
\tag{A.6}
$$

with

$$
V_1 = \int_0^L \left[\frac{p_1}{\lambda_1(w^*)}\exp\left(\frac{\mu x}{\lambda_1(w^*)}\right)(w_1 - w_1^*)^2 + \frac{p_2}{\lambda_2(w^*)}\exp\left(\frac{-\mu x}{\lambda_2(w^*)}\right)(w_2 - w_2^*)^2\right]dx,
\tag{A.7}
$$

$$
V_2 = \int_0^L \left[\frac{p_1}{\lambda_1(w^*)}\exp\left(\frac{\mu x}{\lambda_1(w^*)}\right)(\partial_t w_1)^2 + \frac{p_2}{\lambda_2(w^*)}\exp\left(\frac{-\mu x}{\lambda_2(w^*)}\right)(\partial_t w_2)^2\right]dx,
\tag{A.8}
$$

$$
V_3 = \int_0^L \left[\frac{p_1}{\lambda_1(w^*)}\exp\left(\frac{\mu x}{\lambda_1(w^*)}\right)(\partial_{tt} w_1)^2 + \frac{p_2}{\lambda_2(w^*)}\exp\left(\frac{-\mu x}{\lambda_2(w^*)}\right)(\partial_{tt} w_2)^2\right]dx,
\tag{A.9}
$$

with positive constant coefficients p_1, p_2 and $\mu > 0$. Then, proceeding exactly as [11, Thm. 2.4, Thm. 6.4], (A.4) guarantees the existence of p_1, p_2 and μ such that, along C^2 solutions,

$$
\frac{dV(t)}{dt} \geq \gamma V(t),
\tag{A.10}
$$

for some $\gamma > 0$, which implies that the system is not stable.

Acknowledgement

The authors wish to thank Jean-Michel Coron for his influence in the field of research covered by this paper. They also wish to thank the ANR-Tremplin StarPDE ANR-24-ERCS-0010.

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