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FINITE-TIME OUTPUT REGULATION FOR LINEAR TIME-VARYING HYPERBOLIC BALANCE LAWS*

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Abstract. This work is concerned with the output regulation problem for a nonautonomous infinite-dimensional system. Specifically, a regulator for boundary controlled time-varying hyperbolic systems is designed. The disturbances can act within the space domain, and affect both boundaries and the to-be-controlled output. The to-be-controlled output comprises in-domain pointwise, distributed, and boundary outputs. The output regulation problem is solved in finite time. The regulator design is based on the solvability of the regulator equations. Due to the time-varying setting of the system and the generality of the to-be-controlled output, solving regulator equations becomes more challenging compared to the case of autonomous systems. A novel method is introduced to overcome this difficulty. By considering the regulator equations as a control system, we examine the dual system of the regulator equations and transform the solvability of the regulator equations into the validity of an observability-like inequality. Under the conditions regarding the boundary coupling term and the to-be-controlled output, we have proven this inequality. Additionally, a time-varying setting also brings an advantage to the problem. Since the regulator equations are time-dependent, their solvability does not depend on the eigenmodes of the signal model. On the contrary, in the case of autonomous systems, its solvability depends on the relationship between the plant transfer behavior and the eigenmodes of the signal model.

Key words. hyperbolic systems, nonautonomous systems, output regulation

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1. Introduction.

Control of partial differential equations (PDEs) has garnered significant attention due to their mathematical complexity and applications in various other fields such as engineering and physics. One significant class of PDE systems is hyperbolic systems, which arise in many application scenarios such as open channels, gas flow pipelines, or road traffic flow models. The boundary stabilization of these hyperbolic systems has been considered in the literature for decades; see, for instance, [6]. Therein, the exponential stability of hyperbolic systems is studied. More recently, finite-time stabilization of hyperbolic systems has also received much attention. One can refer to [14, 15] for finite-time stabilization of homogeneous linear and quasilinear hyperbolic systems and to [12] for finite-time stabilization of linear time-varying hyperbolic systems. In [12], a time-dependent backstepping method was used to design the state feedback control.

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In this paper, we investigate the output regulation problem of the hyperbolic systems. For the output regulation problems, unlike the stabilization problems, the objective is to design feedback control such that the output of the system tracks a given reference and rejects the disturbances. There has been a very fruitful literature on the output regulation of the hyperbolic system. In [1, 2], boundary disturbance rejection for linear 2×2 hyperbolic systems was considered by using the backstepping approach. Concerning robust output regulation, [16] used the backstepping method to design a robust state feedback regulator for boundary controlled linear 2×2 hyperbolic systems. Therein, the output to be controlled is assumed to be available for measurement. Therefore, the regulator design is based on the internal model principle. Later on, [19] generalized this work into general $n \times n$ linear heterodirectional hyperbolic systems, where the so-called p-copy internal model principle has to be fulfilled in order to achieve the robust output regulation. In addition to the previous references, we would also like to mention some works on the output regulation of other types of infinite-dimensional systems, including [31] for cascaded network of hyperbolic systems, [23, 27] for the heat equation, [4] for the Korteweg–de Vries equation, [22] for the beam equation, [28] for thermoelastic systems, and [32] for infinite-dimensional nonlinear systems.

Concerning the finite-time output regulation of hyperbolic systems, which is the focus of this article, the first result was obtained in [17], where the backstepping method was used to design the feedback regulator for boundary controlled linear 2×2 time-invariant hyperbolic systems. Moreover, [18, 20] achieved finite-time output regulation for general $n\times n$ time-invariant hyperbolic systems with different convergent time. These three works focused on autonomous hyperbolic systems.

This paper is concerned with the finite-time output regulation problem for linear hyperbolic systems when the coupling coefficients of the system depend on both time and space variables. Therein, the disturbances can act within the domain, affecting both boundaries and the output to be controlled. The output to be controlled comprises in-domain pointwise, distributed, and boundary outputs. In this work, we focus on the design of the feedback regulator, assuming that the system states, reference signal states, and disturbance states are known. Using the results from [12], we transform the design of the feedback regulator into the solvability of regulator equations.

Compared to the literature mentioned above, in particular [17], this paper considers nonautonomous hyperbolic systems, which introduces new challenges to the solvability of regulator equations. As mentioned in [17], the regulator equations of time-independent hyperbolic systems can be expressed as ordinary differential equations (ODEs) and can be explicitly solved. The solvability condition can be characterized as the relationship between the signal model and the transfer behavior of the system. However, under the time-varying setting, the regulator equations are PDEs rather than ODEs. Due to the time-varying setting and the generality of the to-be-controlled output, directly solving the regulator equations becomes difficult. We applied a novel approach to address this challenge. We consider the regulator equations as a control system. Similarly to dealing with controllability problems, we examine the dual system of the regulator equations. Then the solvability of the regulator equations is transformed into the validity of an observability-like inequality, and Lyapunov-like functions are used to prove this observability-like inequality. We characterize the solvability of the regulator equations through the solvability of an operator equation, similarly to how [13] represents null-controllability via an operator equation. In handling the operator equation, we employ a dual approach to analyze the observability of the dual system, whereas [13] utilizes the Fredholm theory.

The Lyapunov-like functions share weights similar to those used in (1.29) of [15], where a more general setting is considered. The weights have been previously proposed in [13]. These weights are crucial to establishing the well-posedness of the broad solutions where the boundary conditions have the form of (2.4) (see the proof of Lemma 3.2 of [13]). These weights are reused in the time-varying setting; see [12]. Through this method, we can only obtain feedback gain function with L^2 regularity over a finite-time domain, which restricts us to solving the output regulation problem only within a finite-time domain and considering only broad solution (with weak regularity) to the system.

In addition, due to our approach, we need assumptions on the dimensions of the system and the to-be-controlled output, namely, the number of equations with negative speeds (i.e., dimension of the input) is not less than the number of equations with positive speeds, which is not less than the dimension of the to-be-controlled output. In the meantime, the time-varying setting also brings the following advantage: the solvability of the regulator equations no longer depends on the relationship between the plant transfer behavior and the eigenmodes of the signal model. In other words, our approach relaxes the assumptions of [17, 18, 20]. Due to the discontinuity in spatial variables of the dual system of the regulator equations, it is necessary to apply specific techniques for the well-posedness. Inspired by the proof in [12, 13, 15], this is done in Appendix A. The proof of the well-posedness involves essentially the weight norms $\|\cdot\|_{\mathcal{B}_1}$ and $\|\cdot\|_{\mathcal{B}_2}$, which have been proposed in [13] and are considered in [15] as well. The proof of the well-posedness is based on the technique in [13, 15], but still needs to be precisely proven, as done in this paper.

The remaining part of this paper is organized as follows. In section 2, we introduce the considered output regulation problem. Some preliminaries needed in the paper is given in section 3. Then section 4 presents the main results of this paper, namely, the design of the finite-time regulator. The well-posedness results for the broad solution and the C^1 solution are provided in Appendices A and B, respectively.

Throughout the paper, we use the following notation. For a domain Ω in $\mathbb{R}^n$, a Banach space X, and any nonnegative integer m, let $C_U^m(\Omega;X)$ denote the vector space consisting of all functions $f:\Omega\to X$ which, together with all their partial derivatives $D^\alpha f$ of orders $|\alpha|\le m$, are bounded and uniformly continuous on Ω. For some constants $T>0$ and $0\le t_0<T$, we define the domain $\mathcal{D}(t_0)=\{(t,x)|t_0<t<T,\, 0<x<1\}$, and we define the function space $\mathcal{B}(t_0)=C^0([t_0,T];L^2(0,1))\cap C^0([0,1];L^2(t_0,T))$. Let l belong to $\mathbb{N}^+$ and let x_i, $i=0,1,\dots,l$, be some points in $[0,1]$ satisfying $0=x_0<x_1<\dots<x_l=1$. We define the domain $\mathcal{D}_l(t_0)=\{(t,x)|t_0<t<T, x\in\cup_{i=1}^l(x_{i-1},x_i)\}$ and define the function space $\mathcal{B}_l(t_0)=C^0([t_0,T];L^2(0,1))\cap C_U^0(\cup_{i=1}^l(x_{i-1},x_i);L^2(t_0,T))$. For a vector ν and a matrix A, denote by $\|\nu\|$ the Euclidean norm and by $\|A\|$ the matrix norm of A associated to the Euclidean norm. For symmetric matrices P and Q, $P>0$ ($P\ge 0$) means that P is positive (nonnegative) definite, and $P>Q$ ($P\ge Q$) means $P-Q>0$ ($P-Q\ge 0$). Denote by Id_n the $n\times n$ identity matrix. Denote by $\mathrm{diag}(A_1,\dots,A_n)$ the block diagonal matrix with matrices $A_1,\dots,A_n$ on the diagonal, where A_i are square matrices of potentially different sizes, and all off-diagonal blocks are zero matrices of appropriate dimensions.

2. Problem statement.

In this paper, combining the systems from [12, 20], we consider the following linear time-varying $n\times n$ hyperbolic system: for (t,x) in $\mathcal{D}(t_0)$,

$$\partial_t w(t,x)+\Lambda(t,x)\partial_x w(t,x)=A(t,x)w(t,x)+g_1(t,x)d(t), \tag{2.1a}$$

$$w_+(t,0)=Q(t)w_-(t,0)+g_2(t)d(t), \tag{2.1b}$$

$$w_-(t,1) = u(t) + g_3(t)d(t), \tag{2.1c}$$
$$w(t_0,x) = w^0(x), \tag{2.1d}$$
$$y(t) = \mathcal{C}_t[w(t,\cdot)] + g_4(t)d(t). \tag{2.1e}$$

In (2.1), $w : \mathcal{D}(t_0) \to \mathbb{R}^n$ is the state, w^0 in $L^2(0,1)^n$ is the initial data at time t_0, $u(t)$ in $\mathbb{R}^m$ is the control input, $d(t)$ in $\mathbb{R}^h$ is the disturbance, and $y(t)$ in $\mathbb{R}^q$ is the output to be controlled. The matrix $A = (a_{ij})_{1\le i,j\le n}$ couples the equations of the system inside the domain, the matrix Q couples the equations of the system on the boundary $x = 0$, and the matrices g_i, $i = 1,\dots,4$, are disturbance input locations. Let us make the following assumptions on all coefficients involved in (2.1).

Assumption 2.1. The matrix Λ is diagonal, namely,

$$\Lambda(t,x) = \operatorname{diag}(\lambda_1(t,x),\dots,\lambda_n(t,x))$$

for every (t,x) in $[0,\infty)\times[0,1]$.

Assumption 2.2. Assume that $n \ge 2$. Denote by m in $\{1,\dots,n-1\}$ the number of equations with negative speeds and by $p = n-m$ in $\{1,\dots,n-1\}$ the number of equations with positive speeds. We assume that there exists some $\varepsilon_0 > 0$ such that for every (t,x) in $[0,\infty)\times[0,1]$, we have

$$\lambda_1(t,x) < \ldots < \lambda_m(t,x) < -\varepsilon_0 < 0 < \varepsilon_0 < \lambda_{m+1}(t,x) < \ldots < \lambda_n(t,x), \tag{2.2}$$

and, for every i in $\{1,\dots,n-1\}$,

$$\lambda_{i+1}(t,x) - \lambda_i(t,x) > \varepsilon_0. \tag{2.3}$$

Assumption 2.2 is identical to the assumption in [12], where the finite-time stabilization problem is considered. Condition (2.2) consists of two components: strict hyperbolicity and the propagation speeds being uniformly bounded away from zero. As stated in [12], the latter is expected for finite-time stabilization, while (2.3) is mainly technical. For more details, one can refer to the examples provided in Remarks 1.9 and 1.10 of [12]. Throughout this paper, for a vector (or vector-valued function) ν in $\mathbb{R}^n$ and a matrix (or matrix-valued function) B in $\mathbb{R}^{n\times n}$, we use the notation

$$\nu = \begin{pmatrix} \nu_- \\ \nu_+ \end{pmatrix}, \quad B = \begin{pmatrix} B_{--} & B_{-+} \\ B_{+-} & B_{++} \end{pmatrix},$$

with v_- in $\mathbb{R}^m$, v_+ in $\mathbb{R}^p$ and B_{--} in $\mathbb{R}^{m\times m}$, B_{-+} in $\mathbb{R}^{m\times p}$, B_{+-} in $\mathbb{R}^{p\times m}$, B_{++} in $\mathbb{R}^{p\times p}$.

Assumption 2.3. The following regularities hold for Λ, A, and Q:

$$\Lambda \in C^1([0,\infty)\times[0,1])^{n\times n}, \quad A \in C^0([0,\infty)\times[0,1])^{n\times n}, \quad Q \in C^0([0,\infty))^{p\times m},$$
$$\Lambda, \partial_x\Lambda, A \in L^\infty((0,\infty)\times(0,1))^{n\times n}, \quad Q \in L^\infty(0,\infty)^{p\times m}.$$

There exist constants $M_0, M_1, M_Q > 0$ such that

$$\|\Lambda\|_{L^\infty((0,\infty)\times(0,1))^{n\times n}} \le M_0, \quad \|\partial_x\Lambda\|_{L^\infty((0,\infty)\times(0,1))^{n\times n}} \le M_0,$$
$$\|A\|_{L^\infty((0,\infty)\times(0,1))^{n\times n}} \le M_1, \quad \|Q\|_{L^\infty(0,\infty)^{p\times m}} \le M_Q.$$

The output to be controlled, $y(t)$ in $\mathbb{R}^q$, is modeled by the formal output operator $\mathcal{C}_t$, which satisfies the following assumption.

Assumption 2.4. Given $f_i = (f_{i-}, f_{i+})$, $i = 0, 1, \ldots, l$, and $c = (c_-, c_+)$ satisfying

$$f_{i-} \in C^0([0,\infty))^{q\times m} \cap L^\infty(0,\infty)^{q\times m}, \quad f_{i+} \in C^0([0,\infty))^{q\times p} \cap L^\infty(0,\infty)^{q\times p},$$
$$c_- \in C^0([0,\infty); L^2(0,1))^{q\times m} \cap L^\infty((0,\infty); L^2(0,1))^{q\times m},$$
$$c_+ \in C^0([0,\infty); L^2(0,1))^{q\times p} \cap L^\infty((0,\infty); L^2(0,1))^{q\times p},$$

for any $t \geq 0$ and $\tilde{n}$ in $\mathbb{N} \setminus \{0\}$, the operator $\mathcal{C}_t$ is defined by

$$\begin{aligned} &\mathcal{C}_t : C^0([0,1])^{n\times\tilde{n}} \to \mathbb{R}^{q\times\tilde{n}} \\ &\bar{\rho} \mapsto \sum_{i=0}^{l} f_i(t)\bar{\rho}(x_i) + \int_0^1 c(t,x)\bar{\rho}(x)\mathrm{d}x. \end{aligned} \tag{2.4}$$

There exist constants $M_f, M_c > 0$ such that

$$\max_{0\leq i\leq l} \|f_i\|_{L^\infty(0,\infty)^{q\times n}} \leq M_f, \quad \|c\|_{L^\infty((0,\infty);L^2(0,1))^{q\times n}} \leq M_c.$$

Clearly, for any $0 \leq t_0 < T$, $\tilde{n}$ in $\mathbb{N} \setminus \{0\}$, and ρ in $\mathcal{B}(t_0)^{n\times\tilde{n}}$, $(t \mapsto \mathcal{C}_t[\rho(t,\cdot)])$ is in $L^2(t_0,T)^{q\times\tilde{n}}$. It comprises in-domain pointwise, distributed, and boundary outputs. It encompasses the outputs used in [17, 18, 20, 31]. These output types are widely applied in real-world problems that can be modeled by hyperbolic systems. For boundary output, one can refer to [6, 7] for the boundary set-point control problem of the Saint-Venant equations. In [5], a heat exchanger with in-domain pointwise output is considered. The boundary conditions having the form of (2.4) have been considered in [13, 15].

The disturbance $d(t)$ is in $\mathbb{R}^h$. The corresponding disturbance input locations satisfy the following assumption.

Assumption 2.5. Matrix-valued functions g_i, $i = 1, 2, 3, 4$, are known and have the following regularities:

$$g_1 \in C^0([0,\infty) \times [0,1])^{n\times h}, \quad g_2 \in C^0([0,\infty))^{p\times h},$$
$$g_3 \in C^0([0,\infty))^{m\times h}, \quad g_4 \in C^0([0,\infty))^{q\times h}.$$

The disturbance $d(t)$ and the reference input $r(t)$ in $\mathbb{R}^q$ to be tracked by the output $y(t)$ are the solutions to the following finite-dimensional signal model: for $t > t_0$,

$$\begin{aligned} &\dot{v}(t) = S(t)v(t), \quad v(t_0) = v^0, \\ &d(t) = p_d(t)v(t), \quad r(t) = p_r(t)v(t), \end{aligned} \tag{2.5}$$

where v^0 is in $\mathbb{R}^{n_v}$. The coefficients of (2.5) satisfy the following assumption.

Assumption 2.6. Matrix-valued functions S, p_d, and p_r are known. $S : [0,\infty) \to \mathbb{R}^{n_v\times n_v}$ is measurable and bounded on every finite subinterval of time, p_d is in $C^0([0,\infty))^{h\times n_v}$, and p_r is in $C^0([0,\infty))^{q\times n_v}$.

By Assumption 2.6, there exists a unique continuous transition matrix $\Psi : [0,\infty)^2 \to \mathbb{R}^{n_v\times n_v}$ of S such that the solution of (2.5) is given by $v(t) = \Psi(t,t_0)v^0$. One can refer to [10, p. 5] for the properties of transition matrix Ψ. Denote by

$$e_y(t) = y(t) - r(t) \tag{2.6}$$

the output tracking error. Inspired by [12, 20], let us give the notion of the uniform finite-time output regulation that we are interested in.

DEFINITION 2.7. *The output y of system* (2.1) *achieves the uniform finite-time output regulation within settling time T_0 if, for any $T > T_0$, there exists a feedback regulator $u = \mathcal{K}_T[w, v]$, such that for all $0 \le t_0 < T - T_0$, w^0 in $L^2(0,1)^n$, and v^0 in $\mathbb{R}^{n_v}$, the output tracking error e_y satisfies $e_y = 0$ a.e. in $(t_0 + T_0, T)$.*

Remark 2.8.

1. Ensuring that the system output tracks a given reference signal is a classic goal in control theory. The output regulation for linear finite-dimensional system is well understood and is well introduced in, for example, [21, 30].
2. The uniformity means that the output regulation is achieved uniformly to the initial time t_0.
3. The output regulation is considered in any finite interval (t_0, T), and the regulator design $\mathcal{K}_T$ is relative to T. This restriction is due to the machinery of proof. See subsection 4.2 for details.

3. Preliminaries on characteristics. In this section, let us introduce some known facts on the characteristics associated with system (2.1) and the entry and exit times for the interval $[x_{i-1}, x_i]$, $i = 1, \dots, l$; see [12]. To this end, we use the extension method introduced in [12] to extend Λ to a function of $\mathbb{R}^2$ (still denoted by Λ) by keeping Assumptions 2.1 to 2.3. For every $j = 1, \dots, n$, let χ_j be the flow associated with λ_j, namely, for every (t, x) in $\mathbb{R}^2$, the function $s \mapsto \chi_j(s; t, x)$ is the solution to the ODE: for s in $\mathbb{R}$,

$$\frac{\partial}{\partial s}\chi_j(s;t,x) = \lambda_j(s, \chi_j(s;t,x)), \quad \chi_j(t;t,x) = x. \tag{3.1}$$

The existence and uniqueness of the solutions to the ODE (3.1) follow the classical theory. Moreover, since λ_j is bounded, the solution is global and has the regularity

$$\chi_j \in C^1(\mathbb{R}^3), \tag{3.2}$$

and, for every (s, t, x) in $\mathbb{R}^3$, we have

$$\partial_t \chi_j(s;t,x) = -\lambda_j(t,x) \mathrm{e}^{\int_t^s \partial_x \lambda_j(\tau, \chi_j(\tau;t,x)) \mathrm{d}\tau}, \ \partial_x \chi_j(s;t,x) = \mathrm{e}^{\int_t^s \partial_x \lambda_j(\tau, \chi_j(\tau;t,x)) \mathrm{d}\tau}. \tag{3.3}$$

Next we introduce the entry and exit times for the interval $[x_{i-1}, x_i]$, $i = 1, \dots, l$. For $j = 1, \dots, n$, t in $\mathbb{R}$, and x in $[0,1]$, let $s_j^{\mathrm{in},i}(t,x)$ and $s_j^{\mathrm{out},i}(t,x)$ be the entry and exit times of the flow $\chi_j(\cdot; t, x)$ inside the interval $[x_{i-1}, x_i]$, namely, the respective unique solutions to

$$\begin{aligned} &\chi_j(s_j^{\mathrm{in},i}(t,x);t,x) = x_i, \quad \chi_j(s_j^{\mathrm{out},i}(t,x);t,x) = x_{i-1} \quad \text{if } j \in \{1, \dots, m\}, \\ &\chi_j(s_j^{\mathrm{in},i}(t,x);t,x) = x_{i-1}, \quad \chi_j(s_j^{\mathrm{out},i}(t,x);t,x) = x_i \quad \text{if } j \in \{m+1, \dots, n\}. \end{aligned} \tag{3.4}$$

The existence and uniqueness of $s_j^{\mathrm{in},i}(t,x)$ and $s_j^{\mathrm{out},i}(t,x)$ are guaranteed by (2.2) in Assumption 2.2. From (3.2) and by the implicit function theorem, we have

$$s_j^{\mathrm{in},i}, s_j^{\mathrm{out},i} \in C^1(\mathbb{R} \times [0,1]), \quad i = 1, \dots, l, \quad j = 1, \dots, n. \tag{3.5}$$

Especially, we denote the entry and exit times for the interval $[0,1]$ as

$$\begin{aligned} &s_j^{\mathrm{in}}(t,x) = s_j^{\mathrm{in},l}(t,x), \quad s_j^{\mathrm{out}}(t,x) = s_j^{\mathrm{out},1}(t,x) \quad \text{if } j \in \{1, \dots, m\}, \\ &s_j^{\mathrm{in}}(t,x) = s_j^{\mathrm{in},1}(t,x), \quad s_j^{\mathrm{out}}(t,x) = s_j^{\mathrm{out},l}(t,x) \quad \text{if } j \in \{m+1, \dots, n\}. \end{aligned} \tag{3.6}$$

Integrating the ODE (3.1) and using (2.2), we have the following bounds for every t in $\mathbb{R}$ and x in $[0,1]$:

$$t - s_j^{\mathrm{in}}(t,x) < \frac{1}{\varepsilon_0}, \quad s_j^{\mathrm{out}}(t,x) - t < \frac{1}{\varepsilon_0}, \quad j = 1,\dots,n. \tag{3.7}$$

Differentiating (3.4) and using (3.3), we see that for $i = 1,\dots,l$,

$$\begin{aligned} \partial_\mu s_j^{\mathrm{in},i}(t,x) &= -\frac{\partial_\mu \chi_j(s_j^{\mathrm{in},i}(t,x);t,x)}{\lambda_j(s_j^{\mathrm{in},i}(t,x),x_i)} \quad \text{if } j \in \{1,\dots,m\}, \\ \partial_\mu s_j^{\mathrm{in},i}(t,x) &= -\frac{\partial_\mu \chi_j(s_j^{\mathrm{in},i}(t,x);t,x)}{\lambda_j(s_j^{\mathrm{in},i}(t,x),x_{i-1})} \quad \text{if } j \in \{m+1,\dots,n\}, \end{aligned} \tag{3.8}$$

with ∂_μ is ∂_t or ∂_x.

4. Finite-time regulator. In this section, we aim to find a finite-time regulator. Let $T > 0$ and $0 \le t_0 < T$. We consider the time-dependent regulator

$$u(t) = k_v(t)v(t) + \int_0^1 k_w(t,x)w(t,x)\mathrm{d}x, \tag{4.1}$$

with feedback gain functions $k_v : (0,T) \to \mathbb{R}^{m\times n_v}$ and $k_w : \mathcal{D}(0) \to \mathbb{R}^{m\times n}$ to be determined later. By applying (4.1) to system (2.1) and taking the signal model (2.5) into account, we have the closed-loop system for (t,x) in $\mathcal{D}(t_0)$:

$$\dot{v}(t) = S(t)v(t), \quad v(t_0) = v^0, \tag{4.2a}$$
$$\partial_t w(t,x) + \Lambda(t,x)\partial_x w(t,x) = A(t,x)w(t,x) + \tilde{g}_1(t,x)v(t), \tag{4.2b}$$
$$w_+(t,0) = Q(t)w_-(t,0) + \tilde{g}_2(t)v(t), \tag{4.2c}$$
$$w_-(t,1) = k_v(t)v(t) + \int_0^1 k_w(t,x)w(t,x)\mathrm{d}x + \tilde{g}_3(t)v(t), \tag{4.2d}$$
$$w(t_0,x) = w^0(x), \tag{4.2e}$$
$$e_y(t) = \mathcal{C}_t[w(t,\cdot)] - (p_r(t) - \tilde{g}_4(t))v(t), \tag{4.2f}$$

where $\tilde{g}_i = g_i p_d$, $i = 1,2,3,4$, and e_y is defined as in (2.6). Similarly to [12], we consider the broad solutions to (4.2b)–(4.2e). The definition of broad solution and the well-posedness of (4.2b)–(4.2e) are given in Appendix A. We have the following well-posedness result for (4.2b)–(4.2e).

THEOREM 4.1. *Let k_w be in $L^\infty(\mathcal{D}(0))^{m\times n}$ and k_v be in $L^2(0,T)^{m\times n_v}$. Under Assumptions* 2.1 *to* 2.3 *and* 2.5*, for every w^0 in $L^2(0,1)^n$ and v in $C^0([t_0,T])^{n_v}$, there exists a unique broad solution w in $\mathcal{B}(t_0)^n$ to the system* (4.2b)–(4.2e).

Theorem 4.1 is a corollary of Theorem A.3 in Appendix A. Let us now state the main result of this paper.

THEOREM 4.2 (finite-time regulator). *Assume that Assumptions* 2.1 *to* 2.6 *hold and assume that there exist positive constants ε_Q and ε_f such that for all $t \ge 0$,*

$$Q(t)Q(t)^\top > \varepsilon_Q \mathrm{Id}_p \tag{4.3}$$

and

$$f_{l+}(t)f_{l+}(t)^\top > \varepsilon_f \mathrm{Id}_q. \tag{4.4}$$

Let the settling time $T_{\mathrm{unif}}(\Lambda)$ be defined by

$$T_{\mathrm{unif}}(\Lambda) = \sup_{t\geq 0}[s_{m+1}^{\mathrm{out}}(s_m^{\mathrm{out}}(t,1),0) - t]. \tag{4.5}$$

Then the output y achieves the uniform finite-time output regulation within settling time $T_{\mathrm{unif}}(\Lambda)$. More precisely, for any $T > T_{\mathrm{unif}}(\Lambda)$, there exist gain functions k_w in $L^\infty(\mathcal{D}(0))^{m\times n}$ and k_v in $L^2(0,T)^{m\times n_v}$ such that for all $0 \leq t_0 \leq T - T_{\mathrm{unif}}(\Lambda)$, w^0 in $L^2(0,1)^n$, and v^0 in $\mathbb{R}^{n_v}$, the output tracking error e_y of closed-loop system (4.2) *satisfies $e_y = 0$ a.e. in $(t_0 + T_{\mathrm{unif}}(\Lambda), T)$.*

Remark 4.3.

1. It follows from conditions (4.3) and (4.4) that the number of equations with negative speeds m (the number of the control input), the number of equations with positive speeds p, and the number of outputs to be controlled q should satisfy $m \geq p \geq q$.
2. When considering the case of a 2×2 hyperbolic system with scalar output, i.e., $m = p = q = 1$, the conditions (4.3) and (4.4) are equivalent to $|Q(t)|^2 \geq \varepsilon_Q > 0$ and $|f_{l+}(t)|^2 \geq \varepsilon_f > 0$ for all $t \geq 0$. If in addition the coefficients of the system do not depend on time, the result of this paper does not recover the result of [17], and vice versa. On the one hand, the example in [17, section 6] (see also the example in item 5 of this remark) shows that the finite-time output regulation problem can be solved when condition (4.4) is not satisfied. On the other hand, Lemma 1 from [17] provides the sufficient and necessary conditions for the existence of a feedback regulator with time-invariant feedback gains. The following example illustrates that, although the conditions of Lemma 1 from [17] are not satisfied, we can still find a feedback regulator with a time-dependent feedback gain. Consider the following 2×2 system: for all (t,x) in $(0,\infty) \times (0,1)$,

$$\partial_t w_1(t,x) - \partial_x w_1(t,x) = 0, \quad \partial_t w_2(t,x) + \partial_x w_2(t,x) = 0, \tag{4.6a}$$

$$w_2(t,0) = w_1(t,0), \quad w_1(t,1) = u(t), \quad w(0,x) = w^0(x), \tag{4.6b}$$

$$y(t) = w_2(t,1) - w_2(t,1/2), \tag{4.6c}$$

and consider the constant reference signal $r(t) = v(t) \equiv v^*$ for some v^* in $\mathbb{R}\backslash\{0\}$. Direct calculation shows that the numerator $N(s)$ of the transfer function of (4.6) from u to y is $N(s) = \mathrm{e}^{-s} - \mathrm{e}^{-s/2}$. The conditions of Lemma 1 from [17] are not satisfied since $N(0) = 0$ and 0 is the eigenvalue of the signal model. However, conditions (4.3) and (4.4) are satisfied, which implies that there exists time-dependent feedback regulator u. Indeed, by the characteristic method, we have that for $t \geq 2$, $y(t) = u(t-2) - u(t-3/2)$. Then $u(t) = -2tv^*$ solves the finite-time output regulation problem. Roughly speaking, the advantage of Theorem 4.2 lies in the fact that the required conditions (4.3) and (4.4) are independent of the signal model.
3. In (4.5), s_m^{out} and s_{m+1}^{out} defined in section 3 are the exit time of the characteristics for the interval $[0,1]$. The settling time $T_{\mathrm{unif}}(\Lambda)$ has been introduced in [12]. The main result of [12] is used in the proof of Theorem 4.2 (see Theorem 4.4 below). Notice that the settling time $T_{\mathrm{unif}}(\Lambda)$ is only related to the propagation speed Λ of the system. Here is an example of 2×2 system to compute the settling time: for all (t,x) in $(0,\infty) \times (0,1)$,

$$\partial_t w_1(t,x) - (1 + \mathrm{e}^{-t})\partial_x w_1(t,x) = 0,$$
$$\partial_t w_2(t,x) + (1 + 0.5\sin(2\pi t))\partial_x w_2(t,x) = 0.$$

Direct calculation shows $\chi_1(s;t,x) = -s + \mathrm{e}^{-s} + t - \mathrm{e}^{-t} + x$ and $\chi_2(s;t,x) = s - \cos(2\pi s)/(4\pi) - t + \cos(2\pi t)/(4\pi) + x$. It is clear that $s_2^{\mathrm{out}}(t,0) = t+1$. Denote $h(t) = s_1^{\mathrm{out}}(t,1) - t$. We have that $h(t)$ solves $\Phi(h(t),t) = 0$, where $\Phi(h,t) = 1 - h + \mathrm{e}^{-h-t} - \mathrm{e}^{-t}$. Taking the derivative of the relation $\Phi(h(t),t) = 0$ and using the fact that $0.5 \leq h(t) \leq 1$, we have that $h'(t) \geq 0$. Thus, h is a bounded nondecreasing function and, consequently $\lim_{t\to\infty} h(t)$ exists and is equal to $\sup_{t\geq 0} h(t)$. Letting $t \to \infty$ in the relation $\Phi(h(t),t) = 0$, we obtain that $\lim_{t\to\infty} h(t) = 1$. Therefore, $T_{\mathrm{unif}}(\Lambda) = 2$.

4. Condition (4.3) is expected for the output regulation to be achieved. Here, we provide an example to illustrate that when condition (4.3) is not satisfied, the output regulation problem may have no solution. Let us consider the following system: for all (t,x) in $(0,\infty)\times(0,1)$,

$$\begin{aligned}
&\partial_t w(t,x) + \Lambda \partial_x w(t,x) = 0,\\
&w_+(t,0) = Q w_-(t,0), \quad w_-(t,1) = u(t),\\
&w(0,x) = w^0(x), \quad y(t) = w_+(t,1),
\end{aligned}$$

where $\Lambda = \mathrm{diag}(-2,-1,1,2)$, $w_- = (w_1,w_2)^\top$, $w_+ = (w_3,w_4)^\top$, and $Q = \left(\begin{smallmatrix}1 & 2\\ 2 & 4\end{smallmatrix}\right)$. Clearly, condition (4.3) is not satisfied. By the characteristic method, for $t > 2$, the explicit representation of the output is

$$y(t) = \begin{pmatrix} w_3(t,1) \\ w_4(t,1) \end{pmatrix} = \begin{pmatrix} u_1(t-3/2) + 2u_2(t-2) \\ 2u_1(t-1) + 4u_2(t-3/2) \end{pmatrix}.$$

We observe that for $t > 2$, $w_4(t-1/2,1) = 2w_3(t,1)$. Then, for the constant signal $r(t) = (1,0)^\top$, the finite-time output regulation cannot be achieved. In the absence of a zero-order term, namely, when $A = 0$, as in this example, [33] proves null-controllability without any assumptions about the structure of Q. For the general case (A is not necessarily zero), it is shown in [3, 11] that one can reach the null-controllability at time $T_{\mathrm{unif}}(\Lambda)$ without any assumption about the structure of Q (indeed, when considering the time-independent case, $T_{\mathrm{unif}}(\Lambda)$ is the same time as in [3, 11]). When Q is required to possess certain specific structures that differ from condition (4.3), [13] proves that null-controllability can be achieved within a smaller control time. In [24], the minimal control time for exact controllability is established when Q is full row rank. For 2×2 hyperbolic systems with $Q = 0$, [25] investigates the minimal null-controllability time. Furthermore, [26] explicitly characterizes the bounds of the minimal null-controllability time for $n \times n$ hyperbolic systems with respect to the zero-order term A. These facts, to some extent, reflect the differences between the null-controllability and the output regulation.

5. Condition (4.4) is mainly technical. This assumption is needed because $f_{l+}(t) f_{l+}(t)^\top > 0$ is necessary for the matrix $\mathcal{P}(t)$ to be positive definite (see in particular (4.28) below). However, this condition is not necessary for the output of some systems to achieve output regulation. Indeed, consider system (4.6a) and (4.6b) again, but with a different output $y(t) = w_2(t,1/2)$. By the characteristic method, we have that for $t > 3/2$, $y(t) = w_2(t,1/2) = u(t-3/2)$. Then for any given reference signal r, the control $u(t) = r(t+3/2)$ enables the finite-time output regulation to be achieved.

Before providing the proof of Theorem 4.2, let us explain its difficulty and how we overcome these difficulties. In Theorem 4.1, we consider the broad solution to system (4.2b)–(4.2e), which has only weak regularity $\mathcal{B}(t_0)^n = C^0([t_0,T];L^2(0,1)^n)\cap C^0([0,1];L^2(t_0,T)^n)$, and we only consider system (4.2b)–(4.2e) over finite-time domain (t_0,T) rather than infinite-time domain (t_0,∞). The reason for these restrictions lies in the time-varying nature of the system, which introduces new challenges in the output regulation problem. In detail, due to the time-varying setting, the regulator equations (see (4.9) below) are PDEs rather than ODEs as in [17]. In [17], the hyperbolic system to be considered is time independent, and thus the solvability condition for the regulator equations can be characterized as the relationship between the signal model and the transfer behavior of the system. Furthermore, the solution to the regulator equations is independent of time, allowing for the design of regulator over infinite-time domain $(0,\infty)$ and the consideration of system over infinite-time domain $(0,\infty)$ as well. However, when considering time-varying systems and accounting for boundary, pointwise as well as distributed outputs (2.4), directly finding solution to the regulator equations becomes challenging. To overcome new difficulties, we examine the dual system of the regulator equations, thereby transforming the issue of solvability of regulator equations over any finite-time domain $(0,T)$ into proving an observability-like inequality (as given in (4.22) below) regarding the solution to the dual system. Then we use Lyapunov-like functions (defined in (4.23) and (4.24) below) to prove observability-like inequality. Through this method, we can only find gain function k_v in $L^2(0,T)^{m\times n_v}$, rather than in more regular function spaces such as $C^0([0,T])^{m\times n_v}$. Furthermore, we cannot extend the gain function k_v to the infinite-time domain $(0,\infty)$. Due to the regularity of gain function k_v, we can only consider system (4.2b)–(4.2e) over finite-time domain (t_0,T) and consider the broad solution w in $\mathcal{B}(t_0)^n$ to system (4.2b)–(4.2e).

In Theorem 4.2, we provide sufficient conditions (4.3) and (4.4) for the solvability of the regulator equations. In detail, (4.3) implies that the boundary coupling coefficient matrix Q is uniformly row full rank, and (4.4) implies that the to-be-controlled output should include $f_{l+}(t)w_+(t,1)$ and f_{l+} is uniformly row full rank. These two conditions ensure that any q-dimensional reference signal can be tracked. We use these conditions in proving the observability-like inequality. These conditions are not required for the output regulation problem in time-independent hyperbolic systems, as mentioned in [17, 18, 20]. They arise from time-varying settings and the machinery of the proof.

Besides, a time-varying setting also brings an advantage to the output regulation problem. Since the regulator equations are time dependent, its solvability no longer relies on the relationship between the signal model and the transfer behavior of the system. An output regulation problem can be achieved for any signal model (2.5).

We prove Theorem 4.2 in two subsections. In subsection 4.1, we remove the dependency of v in (4.2b)–(4.2d) and (4.2f) and provide the feedback gain k_w by using the result in [12]. In subsection 4.2, we prove that the regulator equations admit a solution under the conditions (4.3) and (4.4) and therefore provide the feedback gain k_v.

4.1. Removal of the dependency of v. Let t_0 be in $[0,T-T_{\text{unif}}(\Lambda))$. Inspired by [17], we introduce a bounded invertible change of coordinates to eliminate the dependency of v in (4.2b)–(4.2d) and (4.2f),

$$z(t,x) = w(t,x) - \Pi(t,x)v(t), \tag{4.7}$$

with $\Pi = [\Pi_{ij}] : \mathcal{D}(0) \to \mathbb{R}^{n\times n_v}$. Then (4.2) takes the following form: for (t,x) in $\mathcal{D}(t_0)$,

$$\dot{v}(t) = S(t)v(t), \quad v(t_0) = v^0, \tag{4.8a}$$
$$\partial_t z(t,x) + \Lambda(t,x)\partial_x z(t,x) = A(t,x)z(t,x), \tag{4.8b}$$
$$z_+(t,0) = Q(t)z_-(t,0), \tag{4.8c}$$
$$z_-(t,1) = \int_0^1 k_w(t,x)z(t,x)\mathrm{d}x, \tag{4.8d}$$
$$z(t_0,x) = w^0(x) - \Pi(t_0,x)v^0, \tag{4.8e}$$
$$e_y(t) = \mathcal{C}_t[z(t,\cdot)]; \tag{4.8f}$$

if Π is the solution to the regulator equations: for (t,x) in $\mathcal{D}(0)$,

$$\partial_t\Pi(t,x) + \Lambda(t,x)\partial_x\Pi(t,x) = A(t,x)\Pi(t,x) - \Pi(t,x)S(t) + \tilde{g}_1(t,x), \tag{4.9a}$$
$$\Pi_+(t,0) = Q(t)\Pi_-(t,0) + \tilde{g}_2(t), \tag{4.9b}$$
$$\mathcal{C}_t[\Pi(t,\cdot)] = (p_r(t) - \tilde{g}_4(t)), \tag{4.9c}$$

and

$$k_v(t) = \Pi_-(t,1) - \tilde{g}_3(t) - \int_0^1 k_w(t,x)\Pi(t,x)\mathrm{d}x, \tag{4.10}$$

where $\Pi(t,x) = (\Pi_-^\top, \Pi_+^\top)^\top(t,x)$ with $\Pi_-(t,x)$ in $\mathbb{R}^{m\times n_v}$ and $\Pi_+(t,x)$ in $\mathbb{R}^{p\times n_v}$. The finite-time stability of z-subsystem (4.8b)–(4.8e) follows from the following theorem, which is the main result of [12].

THEOREM 4.4. *Under Assumptions* 2.1 *to* 2.3*, there exists a gain function* k_w *in* $L^\infty(\mathcal{D}(0))^{m\times n}$ *such that for any* w^0 *in* $L^2(0,1)^n$, $\Pi(t_0,\cdot)$ *in* $L^2(0,1)^{n\times n_v}$*, and* v^0 *in* $\mathbb{R}^{n_v}$*, system* (4.8b) *and* (4.8c) *with feedback law* (4.8d) *is finite-time stable with settling time* $T_{\text{unif}}(\Lambda)$ *defined by* (4.5)*.*

Remark 4.5. In [12], k_w can be defined for infinite-time interval $(0,\infty)$. Thus, k_w does not depend on T.

If there exists a solution Π in $\mathcal{B}(0)^{n\times n_v}$ to the regulator equations (4.9), we can define the feedback gain function k_v in $L^2(0,T)^{m\times n_v}$ by (4.10), and Theorem 4.2 is deduced from Theorem 4.4. The remaining thing is to find a solution to the regulator equations (4.9). This is the goal of the next subsection.

4.2. Regulator equations. In this section, we prove that under the assumptions of Theorem 4.2, the regulator equations (4.9) admit a solution. Postmultiply (4.9) by $\Psi(t,0)$, the transition matrix of S, and denote $\hat{\Pi}(t,x) = \Pi(t,x)\Psi(t,0)$. This yields the following equations: for (t,x) in $\mathcal{D}(0)$,

$$\partial_t\hat{\Pi}(t,x) + \Lambda(t,x)\partial_x\hat{\Pi}(t,x) = A(t,x)\hat{\Pi}(t,x) + \hat{g}_1(t,x), \tag{4.11a}$$
$$\hat{\Pi}_+(t,0) = Q(t)\hat{\Pi}_-(t,0) + \hat{g}_2(t), \tag{4.11b}$$
$$\mathcal{C}_t[\hat{\Pi}(t,\cdot)] = \hat{g}_4(t), \tag{4.11c}$$

where $\hat{g}_1(t,x) = \tilde{g}_1(t,x)\Psi(t,0)$, $\hat{g}_2(t) = \tilde{g}_2(t)\Psi(t,0)$, and $\hat{g}_4(t) = (p_r(t) - \tilde{g}_4(t))\Psi(t,0)$.

Remark 4.6. The solvability of (4.11) does not depend either on the signal matrix S or on the initial condition v^0. This property is essentially different from the time-independent case, where an ODE depending on S needs to be solved for Π (see [17]).

The next lemma reduces the solvability of regulator equations (4.11) to the solvability of a homogeneous equation.

LEMMA 4.7. *The regulator equations* (4.11) *have a solution* $\hat{\Pi}$ *in* $\mathcal{B}(0)^{n\times n_v}$ *if, for any* F *in* $L^2(0,T)^q$*, the homogeneous equations, for* (t,x) *in* $\mathcal{D}(0)$,

$$\partial_t\phi(t,x)+\Lambda(t,x)\partial_x\phi(t,x)=A(t,x)\phi(t,x), \tag{4.12a}$$

$$\phi_+(t,0)=Q(t)\phi_-(t,0), \tag{4.12b}$$

$$\mathcal{C}_t[\phi(t,\cdot)]=F(t), \tag{4.12c}$$

admit a solution ϕ *in* $\mathcal{B}(0)^n$.

Proof. For $i=1,\dots,n_v$, denote by $\Pi^i(t,x)$ in $\mathbb{R}^n$ the broad solution to the following equations: for (t,x) in $\mathcal{D}(0)$,

$$\partial_t\Pi^i(t,x)+\Lambda(t,x)\partial_x\Pi^i(t,x)=A(t,x)\Pi^i(t,x)+\hat{g}_1^i(t,x), \tag{4.13a}$$

$$\Pi^i_+(t,0)=Q(t)\Pi^i_-(t,0)+\hat{g}_2^i(t), \tag{4.13b}$$

$$\Pi^i_-(t,1)=0, \tag{4.13c}$$

$$\Pi^i(0,x)=0, \tag{4.13d}$$

where $\hat{g}_1^i$ and $\hat{g}_2^i$ are the ith columns of $\hat{g}_1$ and $\hat{g}_2$, respectively. Due to the well-posedness results (see Theorem A.3), there exists a unique broad solution Π^i in $\mathcal{B}(0)^n$ to system (4.13). For $i=1,\dots,n_v$, denote by ϕ^i the solution to (4.12) with $F(t)=\hat{g}_4^i(t)-\mathcal{C}_t[\Pi^i(t,\cdot)]$, where $\hat{g}_4^i$ is the ith column of $\hat{g}_4$. Thus, $\hat{\Pi}=(\Pi^1+\phi^1,\Pi^2+\phi^2,\dots,\Pi^{n_v}+\phi^{n_v})$ is the solution to (4.11). ∎

Now we prove that under the assumptions of Theorem 4.2, the homogeneous equations (4.12) admit a solution. By the well-posedness results (see Theorem A.3), (4.12a) and (4.12b) together with the initial and boundary conditions

$$\phi_-(t,1)=u^0(t),\quad \phi(0,x)=\phi^0(x) \tag{4.14}$$

have a unique broad solution ϕ in $\mathcal{B}(0)^n$, where u^0 belongs to $L^2(0,T)^m$ and ϕ^0 belongs to $L^2(0,1)^n$. Then define the map $\mathcal{F}_T$ as follows:

$$\begin{array}{llcl}\mathcal{F}_T: & L^2(0,1)^n\times L^2(0,T)^m & \to & L^2(0,T)^q\\ & (\phi^0,u^0) & \mapsto & (t\mapsto \mathcal{C}_t[\phi(t,\cdot)]),\end{array} \tag{4.15}$$

where ϕ in $\mathcal{B}(0)^n$ is the broad solution to (4.14), (4.12a), and (4.12b). It follows that $\mathcal{F}_T$ is a linear continuous map from $L^2(0,1)^n\times L^2(0,T)^m$ into $L^2(0,T)^q$.

We get that the homogeneous regulator equations (4.12) have a solution if the map $\mathcal{F}_T$ is onto. In order to decide whether $\mathcal{F}_T$ is onto or not, we use the following classical result of functional analysis (see Theorem 4.13 of [29, p. 100]).

PROPOSITION 4.8. *Let* H_1 *and* H_2 *be two Hilbert spaces. Let* $\mathcal{F}$ *be a linear continuous map from* H_1 *into* H_2*. Then* $\mathcal{F}$ *is onto if and only if there exists* $c>0$ *such that*

$$\|\mathcal{F}^*(\rho)\|_{H_1}\geq c\|\rho\|_{H_2}\quad \forall\rho\in H_2, \tag{4.16}$$

where $\mathcal{F}^*$ *is the adjoint operator of* $\mathcal{F}$.

In order to apply this proposition, we make explicit $\mathcal{F}_T^*$ in the following lemma.

LEMMA 4.9. *Let ω be in $L^2(0,T)^q$. Let θ in $\mathcal{B}_l(0)^n$ be the unique broad solution to the following equations (see Theorem* A.3 *for the well-posedness): for (t,x) in $\mathcal{D}_l(0)$,*

$$\partial_t\theta(t,x)+\partial_x(\Lambda(t,x)\theta(t,x))=-A(t,x)^\top\theta(t,x)-c(t,x)^\top\omega(t), \tag{4.17a}$$

$$\theta_-(t,x_i^+)=\theta_-(t,x_i^-)-\Lambda_{--}(t,x_i)^{-1}f_{i-}(t)^\top\omega(t),\quad i=1,\dots,l-1, \tag{4.17b}$$

$$\theta_-(t,0)=-\Lambda_{--}(t,0)^{-1}[Q(t)^\top\Lambda_{++}(t,0)\theta_+(t,0)+(f_{0+}(t)Q(t)+f_{0-}(t))^\top\omega(t)], \tag{4.17c}$$

$$\theta_+(t,x_i^-)=\theta_+(t,x_i^+)+\Lambda_{++}(t,x_i)^{-1}f_{i+}(t)^\top\omega(t),\quad i=1,\dots,l-1, \tag{4.17d}$$

$$\theta_+(t,1)=\Lambda_{++}(t,1)^{-1}f_{l+}(t)^\top\omega(t), \tag{4.17e}$$

$$\theta(T,x)=0. \tag{4.17f}$$

Then

$$\mathcal{F}_T^*(\omega)=(\theta(0,\cdot),f_{l-}^\top\omega-\Lambda_{--}(\cdot,1)\theta_-(\cdot,1)). \tag{4.18}$$

Proof. Let us first assume that (Λ,A,Q,c,ω) is in $C^2(\overline{\mathcal{D}(0)})^{n\times n}\times C^1(\overline{\mathcal{D}(0)})^{n\times n}\times C^1[0,T]^{p\times m}\times C^1_U(\mathcal{D}_l(0))^{q\times n}\times C^1[0,T]^q$, f_i is in $C^1[0,T]^{q\times n}$, $i=0,1,\dots,l$, and the compatibility conditions

$$\omega(T)=0,\quad \omega'(T)=0 \tag{4.19}$$

hold. Let ϕ^0 in $C^1[0,1]^n$ and u^0 in $C^1[0,T]^m$ be such that

$$\begin{aligned}
&\phi^0_+(0)=Q(0)\phi^0_-(0),\quad \phi^0_-(1)=u^0(0),\\
&(u^0)'(0)=-\Lambda_{--}(0,1)(\phi^0_-)'(1)+A_{-+}(0,1)\phi^0_+(1)+A_{--}(0,1)\phi^0_-(1),\\
&\qquad -\Lambda_{++}(0,0)(\phi^0_+)'(0)+A_{++}(0,0)\phi^0_+(0)+A_{+-}(0,0)\phi^0_-(0)\\
&\qquad =Q(t)[-\Lambda_{--}(0,0)(\phi^0_-)'(0)+A_{-+}(0,0)\phi^0_+(0)+A_{--}(0,0)\phi^0_-(0)]+Q'(t)\phi^0_-(0).
\end{aligned} \tag{4.20}$$

Let ϕ in $C^1(\overline{\mathcal{D}(0)})^n$ be the C^1 solution to (4.14), (4.12a), and (4.12b) (see Theorem B.2). Considering the boundary condition (4.12b) and the definition of the output in (2.4), we have that

$$\begin{aligned}
\mathcal{F}_T(\phi^0,u^0)(t)=\mathcal{C}_t[\phi(t,\cdot)]&=\sum_{i=0}^{l}f_i(t)\phi(t,x_i)+\int_0^1 c(t,x)\phi(t,x)\mathrm{d}x\\
&=(f_{0+}(t)Q(t)+f_{0-}(t))\phi_-(t,0)+\sum_{i=1}^{l}f_i(t)\phi(t,x_i)\\
&\quad+\int_0^1 c(t,x)\phi(t,x)\mathrm{d}x.
\end{aligned} \tag{4.21}$$

Let θ in $C^1_U(\mathcal{D}_l(0))^n$ be the C^1 solution to (4.17) (see Theorem B.2). Then from (4.14), (4.17), (4.12a), and (4.12b), we obtain that, using integrations by parts,

$$
\begin{aligned}
0 &= \sum_{i=1}^{l} \int_0^T \int_{x_{i-1}}^{x_i} \theta(t,x)^\top [\partial_t \phi(t,x) + \Lambda(t,x)\partial_x \phi(t,x) - A(t,x)\phi(t,x)] \mathrm{d}x\mathrm{d}t \\
&= \sum_{i=1}^{l} \Bigg\{ - \int_0^T \int_{x_{i-1}}^{x_i} [\partial_t \theta(t,x) + \partial_x(\Lambda(t,x)\theta(t,x)) + A(t,x)^\top \theta(t,x)]^\top \phi(t,x) \mathrm{d}x\mathrm{d}t \\
&\quad + \int_{x_{i-1}}^{x_i} [\theta(T,x)^\top \phi(T,x) - \theta(0,x)^\top \phi(0,x)] \mathrm{d}x \\
&\quad + \int_0^T [\theta(t,x_i^-)^\top \Lambda(t,x_i)\phi(t,x_i) - \theta(t,x_{i-1}^+)^\top \Lambda(t,x_{i-1})\phi(t,x_{i-1})] \mathrm{d}t \Bigg\} \\
&= \int_0^T \int_0^1 \omega(t)^\top c(t,x)\phi(t,x)\mathrm{d}x\mathrm{d}t - \int_0^1 \theta(0,x)^\top \phi^0(x)\mathrm{d}x \\
&\quad + \int_0^T \omega(t)^\top \left[(f_{0+}(t)Q(t) + f_{0-}(t))\phi_-(t,0) + \sum_{i=1}^{l} f_i(t)\phi(t,x_i) \right] \mathrm{d}t \\
&\quad - \int_0^T (f_{l-}(t)^\top \omega(t) - \Lambda_{--}(t,1)\theta_-(t,1))^\top u^0(t)\mathrm{d}t.
\end{aligned}
$$

Consequently, it follows from (4.21) that

$$
\begin{aligned}
\int_0^T \omega(t)^\top \mathcal{F}_T(\phi^0,u^0)(t)\mathrm{d}t &= \int_0^T \omega(t)\mathcal{C}_t[\phi(t,\cdot)]\mathrm{d}t \\
&= \int_0^1 \theta(0,x)^\top \phi^0(x)\mathrm{d}x + \int_0^T (f_{l-}(t)^\top \omega(t) - \Lambda_{--}(t,1)\theta_-(t,1))^\top u^0(t)\mathrm{d}t,
\end{aligned}
$$

which, together with Claim B.3, concludes the proof of Lemma 4.9. ∎

In next lemma, we prove that under the assumptions of Theorem 4.2, inequality (4.16) holds with respect to operator (4.18).

LEMMA 4.10. *Let the assumptions of Theorem 4.2 hold. Let ω belong to $L^2(0,T)^q$ and θ in $\mathcal{B}_l(0)^n$ be the broad solution to (4.17). Then there exists a constant $c_T > 0$ such that*

(4.22)
$$
\int_0^1 \|\theta(0,x)\|^2\mathrm{d}x + \int_0^T \|f_{l-}(t)^\top \omega(t) - \Lambda_{--}(t,1)\theta_-(t,1)\|^2\mathrm{d}t \geq c_T \int_0^T \|\omega(t)\|^2\mathrm{d}t.
$$

Proof. Let us first assume that $(\Lambda, A, Q, c, \omega)$ is in $C^2(\overline{\mathcal{D}(0)})^{n\times n} \times C^1(\overline{\mathcal{D}(0)})^{n\times n} \times C^1[0,T]^{p\times m} \times C^1_U(\mathcal{D}_l(0))^{q\times n} \times C^1[0,T]^q$, f_i is in $C^1[0,T]^{q\times n}$, $i = 0,1,\ldots,l$, and the compatibility conditions (4.19) hold. Let θ in $C^1_U(\mathcal{D}_l(0))^n$ be the C^1 solution to (4.17).

For $i = 1,\ldots,l$ and $0 \leq t \leq T$, let

$$
V_{i+}(t) = \mathrm{e}^{-Lt} \int_{x_{i-1}}^{x_i} \mathrm{e}^{\alpha_i (x - x_{i-1})} \|\theta_+(t,x)\|^2 \mathrm{d}x, \tag{4.23}
$$

$$
V_{i-}(t) = \mathrm{e}^{-Lt} \int_{x_{i-1}}^{x_i} \mathrm{e}^{\beta_i (x_i - x)} \|\theta_-(t,x)\|^2 \mathrm{d}x, \tag{4.24}
$$

with positive coefficients L, α_i, and β_i to be chosen later. Denote $V(t) = \sum_{i=1}^{l} (V_{i+}(t) + V_{i-}(t))$. The proof of (4.22) is based on identity $V(0) = -\int_0^T \frac{\mathrm{d}V}{\mathrm{d}t}(t)\mathrm{d}t$, and the main idea is as follows. First, $V(0)$ is equivalent to $\int_0^1 \|\theta(0,x)\|^2\mathrm{d}x$. Next, we use integration by parts to express term $\beta_0 \int_0^T \|f_{l-}(t)^\top \omega(t) - \Lambda_{--}(t,1)\theta_-(t,1)\|^2\mathrm{d}t - \int_0^T \frac{\mathrm{d}V}{\mathrm{d}t}(t)\mathrm{d}t$

as a quadratic form. Finally, by applying conditions (4.3) and (4.4) and selecting appropriate constants L, α_i, and β_i, we ensure that this quadratic form is greater than or equal to $c_T\int_0^T\|\omega(t)\|^2\mathrm{d}t$. The weights of the Lyapunov-like functions (4.23) and (4.24) are similar to those used in [12, 13, 15]. In [12, 13] the weights are crucial to establishing the well-posedness of the broad solutions.

Let us proceed with the proof. The time derivative of $V_{i+}(t)$ along the C^1 solution θ to (4.17) is

$$\begin{aligned}\frac{\mathrm{d}V_{i+}(t)}{\mathrm{d}t} &= \mathrm{e}^{-Lt}\int_{x_{i-1}}^{x_i}\mathrm{e}^{\alpha_i(x-x_{i-1})}\theta_+(t,x)^\top[2\partial_t\theta_+(t,x)-L\theta_+(t,x)]\mathrm{d}x\\
&= \mathrm{e}^{-Lt}\int_{x_{i-1}}^{x_i}\mathrm{e}^{\alpha_i(x-x_{i-1})}\theta_+(t,x)^\top[-L\theta_+(t,x)-2\partial_x(\Lambda_{++}(t,x)\theta_+(t,x))\\
&\quad -2A_{++}(t,x)^\top\theta_+(t,x)-2A_{-+}(t,x)^\top\theta_-(t,x)-2c_+(t,x)^\top\omega(t)]\mathrm{d}x\\
&= \mathrm{e}^{-Lt}\Bigg\{\int_{x_{i-1}}^{x_i}\mathrm{e}^{\alpha_i(x-x_{i-1})}\theta_+(t,x)^\top[-(L\mathrm{Id}_p-\alpha_i\Lambda_{++}(t,x)+\partial_x\Lambda_{++}(t,x)\\
&\quad +2A_{++}(t,x)^\top)\theta_+(t,x)-2A_{-+}(t,x)^\top\theta_-(t,x)-2c_+(t,x)^\top\omega(t)]\mathrm{d}x\\
&\quad -\mathrm{e}^{\alpha_i(x_i-x_{i-1})}\theta_+(t,x_i^-)^\top\Lambda_{++}(t,x_i)\theta_+(t,x_i^-)\\
&\quad +\theta_+(t,x_{i-1}^+)^\top\Lambda_{++}(t,x_{i-1})\theta_+(t,x_{i-1}^+)\Bigg\}.\end{aligned}$$

Similarly, the time derivative of $V_{i-}(t)$ along the C^1 solution θ to (4.17) is

$$\begin{aligned}\frac{\mathrm{d}V_{i-}(t)}{\mathrm{d}t} &= \mathrm{e}^{-Lt}\Bigg\{\int_{x_{i-1}}^{x_i}\mathrm{e}^{\beta_i(x_i-x)}\theta_-(t,x)^\top[-(L\mathrm{Id}_m+\beta_i\Lambda_{--}(t,x)+\partial_x\Lambda_{--}(t,x)\\
&\quad +2A_{--}(t,x)^\top)\theta_-(t,x)-2A_{+-}(t,x)\theta_+(t,x)-2c_-(t,x)^\top\omega(t)]\mathrm{d}x\\
&\quad -\theta_-(t,x_i^-)^\top\Lambda_{--}(t,x_i)\theta_-(t,x_i^-)\\
&\quad +\mathrm{e}^{\beta_i(x_i-x_{i-1})}\theta_-(t,x_{i-1}^+)^\top\Lambda_{--}(t,x_{i-1})\theta_-(t,x_{i-1}^+)\Bigg\}.\end{aligned}$$

Taking boundary and jump conditions (4.17b)–(4.17e) into account, we conclude that

(4.25)

$$\begin{aligned}&\mathrm{e}^{Lt}\frac{\mathrm{d}V(t)}{\mathrm{d}t}\\
&=\sum_{i=1}^{l}\int_{x_{i-1}}^{x_i}[-2(\mathrm{e}^{\alpha_i(x-x_{i-1})}c_+(t,x)\theta_+(t,x)+\mathrm{e}^{\beta_i(x_i-x)}c_-(t,x)\theta_-(t,x))^\top\omega(t)\\
&\quad -\mathrm{e}^{\alpha_i(x-x_{i-1})}\theta_+(t,x)^\top(L\mathrm{Id}_p-\alpha_i\Lambda_{++}(t,x)+\partial_x\Lambda_{++}(t,x)+2A_{++}(t,x)^\top)\theta_+(t,x)\\
&\quad -\mathrm{e}^{\beta_i(x_i-x)}\theta_-(t,x)^\top(L\mathrm{Id}_m+\beta_i\Lambda_{--}(t,x)+\partial_x\Lambda_{--}(t,x)+2A_{--}(t,x)^\top)\theta_-(t,x)\\
&\quad -2\theta_+(t,x)^\top(\mathrm{e}^{\alpha_i(x-x_{i-1})}A_{-+}(t,x)^\top+\mathrm{e}^{\beta_i(x_i-x)}A_{+-}(t,x))\theta_-(t,x)]\mathrm{d}x\\
&\quad -\sum_{i=1}^{l-1}\{(\mathrm{e}^{\alpha_i(x_i-x_{i-1})}-1)\theta_+(t,x_i^+)^\top\Lambda_{++}(t,x_i)\theta_+(t,x_i^+)\\
&\quad +\mathrm{e}^{\alpha_i(x_i-x_{i-1})}[2\theta_+(t,x_i^+)^\top f_{i+}(t)^\top\omega(t)+\omega(t)^\top f_{i+}(t)\Lambda_{++}(t,x_i)^{-1}f_{i+}(t)^\top\omega(t)]\\
&\quad -(\mathrm{e}^{\beta_{i+1}(x_{i+1}-x_i)}-1)\theta_-(t,x_i^-)^\top\Lambda_{--}(t,x_i)\theta_-(t,x_i^-)\\
&\quad +\mathrm{e}^{\beta_{i+1}(x_{i+1}-x_i)}[2\theta_-(t,x_i^-)^\top f_{i-}(t)^\top\omega(t)-\omega(t)^\top f_{i-}(t)\Lambda_{--}(t,x_i)^{-1}f_{i-}(t)^\top\omega(t)]\}\end{aligned}$$

$$
\begin{aligned}
&+\theta_+(t,0)^\top\Lambda_{++}(t,0)(\mathrm{e}^{\beta_1 x_1}Q(t)\Lambda_{--}(t,0)^{-1}Q(t)^\top+\Lambda_{++}(t,0)^{-1})\Lambda_{++}(t,0)\theta_+(t,0)\\
&+2\mathrm{e}^{\beta_1 x_1}\theta_+(t,0)^\top\Lambda_{++}(t,0)Q(t)\Lambda_{--}(t,0)^{-1}(f_{0+}(t)Q(t)+f_{0-}(t))^\top\omega(t)\\
&+\mathrm{e}^{\beta_1 x_1}\omega(t)^\top(f_{0+}(t)Q(t)+f_{0-}(t))\Lambda_{--}(t,0)^{-1}(f_{0+}(t)Q(t)+f_{0-}(t))^\top\omega(t)\\
&-\mathrm{e}^{\alpha_l(1-x_{l-1})}\omega(t)^\top f_{l+}(t)\Lambda_{++}(t,1)^{-1}f_{l+}(t)^\top\omega(t)-\theta_-(t,1)^\top\Lambda_{--}(t,1)\theta_-(t,1).
\end{aligned}
$$

Multiply (4.25) by $-\mathrm{e}^{-Lt}$ and integrate over $(0,T)$, and add

$$\beta_0\int_0^T\|f_{l-}(t)^\top\omega(t)-\Lambda_{--}(t,1)\theta_-(t,1)\|^2\mathrm{d}t$$

to both sides of (4.25) with positive coefficient β_0 to be chosen later. Recall $x_0=0$ and $x_l=1$. It follows from (4.17f) that

$$V(0)+\beta_0\int_0^T\|f_{l-}(t)^\top\omega(t)-\Lambda_{--}(t,1)\theta_-(t,1)\|^2\mathrm{d}t=R_1+R_2, \tag{4.26}$$

where

(4.27)

$$
\begin{aligned}
R_1=\int_0^T\mathrm{e}^{-Lt}\sum_{i=1}^{l}\int_{x_{i-1}}^{x_i}&[2(\mathrm{e}^{\alpha_i(x-x_{i-1})}c_+(t,x)\theta_+(t,x)+\mathrm{e}^{\beta_i(x_i-x)}c_-(t,x)\theta_-(t,x))^\top\omega(t)\\
&+\mathrm{e}^{\alpha_i(x-x_{i-1})}\theta_+(t,x)^\top(L\mathrm{Id}_p-\alpha_i\Lambda_{++}(t,x)+\partial_x\Lambda_{++}(t,x)+2A_{++}(t,x)^\top)\theta_+(t,x)\\
&+\mathrm{e}^{\beta_i(x_i-x)}\theta_-(t,x)^\top(L\mathrm{Id}_m+\beta_i\Lambda_{--}(t,x)+\partial_x\Lambda_{--}(t,x)+2A_{--}(t,x)^\top)\theta_-(t,x)\\
&+2\theta_+(t,x)^\top(\mathrm{e}^{\alpha_i(x-x_{i-1})}A_{-+}(t,x)^\top+\mathrm{e}^{\beta_i(x_i-x)}A_{+-}(t,x))\theta_-(t,x)]\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

and

$$R_2=\int_0^T\mathrm{e}^{-Lt}\Theta(t)^\top\mathcal{P}(t)\Theta(t)\mathrm{d}t, \tag{4.28}$$

with

$$
\begin{aligned}
&\Theta(t)=\begin{pmatrix}\Theta_-(t)\\\Theta_+(t)\\\omega(t)\end{pmatrix},\quad \mathcal{P}(t)=\begin{pmatrix}\mathcal{P}_-(t)&0&\mathcal{F}_-(t)\\0&\mathcal{P}_+(t)&\mathcal{F}_+(t)\\\mathcal{F}_-(t)^\top&\mathcal{F}_+(t)^\top&P_\omega(t)\end{pmatrix},\\
&\Theta_-(t)=(\theta_-(t,x_1^-)^\top,\dots,\theta_-(t,x_l^-)^\top)^\top,\quad \Theta_+(t)=(\theta_+(t,x_0^+)^\top,\dots,\theta_+(t,x_{l-1}^+)^\top)^\top,\\
&\mathcal{P}_-(t)=\mathrm{diag}(P_{1-}(t),\dots,P_{l-}(t)),\quad \mathcal{P}_+(t)=\mathrm{diag}(P_{0+}(t),\dots,P_{(l-1)+}(t)),\\
&\mathcal{F}_-(t)=(F_{1-}(t)^\top,\dots,F_{l-}(t)^\top)^\top,\quad \mathcal{F}_+(t)=(F_{0+}(t)^\top,\dots,F_{(l-1)+}(t)^\top)^\top,\\
&P_{i+}(t)=(\mathrm{e}^{\alpha_i(x_i-x_{i-1})}-1)\Lambda_{++}(t,x_i),\quad P_{i-}(t)=-(\mathrm{e}^{\beta_{i+1}(x_{i+1}-x_i)}-1)\Lambda_{--}(t,x_i),\\
&F_{i+}(t)=\mathrm{e}^{\alpha_i(x_i-x_{i-1})}f_{i+}(t)^\top,\quad F_{i-}(t)=\mathrm{e}^{\beta_{i+1}(x_{i+1}-x_i)}f_{i-}(t)^\top,\quad i=1,\dots,l-1,\\
&P_{0+}(t)=-\Lambda_{++}(t,0)(\mathrm{e}^{\beta_1x_1}Q(t)\Lambda_{--}(t,0)^{-1}Q(t)^\top+\Lambda_{++}(t,0)^{-1})\Lambda_{++}(t,0),\\
&F_{0+}(t)=-\mathrm{e}^{\beta_1x_1}\Lambda_{++}(t,0)Q(t)\Lambda_{--}(t,0)^{-1}(f_{0+}(t)Q(t)+f_{0-}(t))^\top,\\
&P_{l-}(t)=\beta_0\Lambda_{--}(t,1)^2+\Lambda_{--}(t,1),\quad F_{l-}(t)=-\beta_0\Lambda_{--}(t,1)f_{l-}(t)^\top,\\
&P_\omega(t)=\beta_0f_{l-}(t)f_{l-}(t)^\top+\sum_{i=1}^{l-1}\Big[\mathrm{e}^{\alpha_i(x_i-x_{i-1})}f_{i+}(t)\Lambda_{++}(t,x_i)^{-1}f_{i+}(t)^\top\\
&\qquad-\mathrm{e}^{\beta_{i+1}(x_{i+1}-x_i)}f_{i-}(t)\Lambda_{--}(t,x_i)^{-1}f_{i-}(t)^\top\Big]\\
&\qquad+\mathrm{e}^{\alpha_l(1-x_{l-1})}f_{l+}(t)\Lambda_{++}(t,1)^{-1}f_{l+}(t)^\top\\
&\qquad-\mathrm{e}^{\beta_1x_1}(f_{0+}(t)Q(t)+f_{0-}(t))\Lambda_{--}(t,0)^{-1}(f_{0+}(t)Q(t)+f_{0-}(t))^\top.
\end{aligned}
$$

Considering the left-hand side of (4.26), we have

$$
\begin{aligned}
&V(0)+\beta_0\int_0^T \|f_{l-}(t)^\top\omega(t)-\Lambda_{--}(t,1)\theta_-(t,1)\|^2\mathrm{d}t\\
&\leq \tilde{M}\left(\int_0^1\|\theta(0,x)\|^2\mathrm{d}x+\int_0^T\|f_{l-}(t)^\top\omega(t)-\Lambda_{--}(t,1)\theta_-(t,1)\|^2\mathrm{d}t\right),
\end{aligned}\tag{4.29}
$$

where

$$
\tilde{M}=\max\left\{\beta_0,\max_{i=1,\dots,l}\left\{\mathrm{e}^{\alpha_i},\mathrm{e}^{\beta_i}\right\}\right\}.\tag{4.30}
$$

Now we deal with the right-hand side of (4.26). Our aim is to choose suitable constants L, β_0, α_i, β_i, $i=1,\dots,l$, such that $R_1+R_2\geq\int_0^T\varepsilon\mathrm{e}^{-Lt}\|\omega(t)\|^2\mathrm{d}t$ for some positive constant ε. Let us first deal with R_2. For any $\varepsilon^*>0$, let β_0, α_i, β_i, $i=1,\dots,l$, be large enough such that

$$
\begin{aligned}
&\beta_0\varepsilon_0>1,\quad \mathrm{e}^{\beta_1x_1}\varepsilon_0\varepsilon_Q>M_0,\quad \alpha_i>0,\quad \beta_{i+1}>0,\quad i=1,\dots,l-1,\\
&\mathrm{e}^{\alpha_l(1-x_{l-1})}\frac{\varepsilon_f}{M_0}\geq\varepsilon^*+\frac{M_f^2}{\varepsilon_0}\sum_{i=0}^{l-1}\left(\frac{\mathrm{e}^{\alpha_i(x_i-x_{i-1})}}{\mathrm{e}^{\alpha_i(x_i-x_{i-1})}-1}+\frac{\mathrm{e}^{\beta_{i+1}(x_{i+1}-x_i)}}{\mathrm{e}^{\beta_{i+1}(x_{i+1}-x_i)}-1}\right)\\
&\quad+\frac{\beta_0M_f^2}{\beta_0\varepsilon_0-1}+2\mathrm{e}^{\beta_1x_1}M_f^2(1+mpM_Q^2)\frac{\mathrm{e}^{\beta_1x_1}mpM_0M_Q^2\varepsilon_0^{-1}-\mathrm{e}^{\beta_1x_1}\varepsilon_Q+M_0\varepsilon_0^{-1}}{\mathrm{e}^{\beta_1x_1}\varepsilon_0\varepsilon_Q-M_0},
\end{aligned}
$$

where ε_0 is defined as in Assumption 2.2, ε_Q is defined as in (4.3), ε_f is defined as in (4.4), M_0 and M_Q are defined as in Assumption 2.3, and M_f is defined as in Assumption 2.4. Direct calculation shows that for all t in $[0,T]$,

$$
\begin{aligned}
&P_{i+}(t)>0,\quad i=0,\dots,l-1,\quad P_{i-}(t)>0,\quad i=1,\dots,l,\\
&P_\omega(t)\geq\varepsilon^*\mathrm{Id}_q+\sum_{i=0}^{l-1}F_{i+}(t)^\top P_{i+}(t)^{-1}F_{i+}(t)+\sum_{i=1}^{l}F_{i-}(t)^\top P_{i-}(t)^{-1}F_{i-}(t).
\end{aligned}\tag{4.31}
$$

Note that (4.3) and (4.4) are necessary for P_{0+} and P_ω to be positive definite, respectively. It follows from (4.31) and the Schur complement lemma (see [8, Appendix 5.5]) that for all t in $[0,T]$, $\Theta(t)^\top\mathcal{P}(t)\Theta(t)\geq\varepsilon^*\|\omega(t)\|^2$, and thus $R_2\geq\int_0^T\varepsilon^*\mathrm{e}^{-Lt}\|\omega(t)\|^2\mathrm{d}t$.

Now let us estimate R_1. For L large enough, we have

$$
\begin{aligned}
R_1\geq&\int_0^T\mathrm{e}^{-Lt}\sum_{i=1}^l\int_{x_{i-1}}^{x_i}[\mathrm{e}^{\alpha_i(x-x_{i-1})}(L-(\alpha_i+1)M_0-2pM_1)\|\theta_+(t,x)\|^2\\
&+\mathrm{e}^{\beta_i(x_i-x)}(L-(\beta_i+1)M_0-2mM_1)\|\theta_-(t,x)\|^2\\
&-(\mathrm{e}^{\alpha_i(x-x_{i-1})}+\mathrm{e}^{\beta_i(x_i-x)})M_1(m\|\theta_+(t,x)\|^2+p\|\theta_-(t,x)\|^2)\\
&+2(\mathrm{e}^{\alpha_i(x-x_{i-1})}c_+(t,x)\theta_+(t,x)+\mathrm{e}^{\beta_i(x_i-x)}c_-(t,x)\theta_-(t,x))^\top\omega(t)]\mathrm{d}x\mathrm{d}t\\
\geq&\int_0^T\mathrm{e}^{-Lt}\sum_{i=1}^l\int_{x_{i-1}}^{x_i}[(L-(\tilde{M}+1)M_0\tilde{M}-2nM_1\tilde{M})(\|\theta_+(t,x)\|^2+\|\theta_-(t,x)\|^2)\\
&+2(\mathrm{e}^{\alpha_i(x-x_{i-1})}c_+(t,x)\theta_+(t,x)+\mathrm{e}^{\beta_i(x_i-x)}c_-(t,x)\theta_-(t,x))^\top\omega(t)]\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

where M_1 is defined as in Assumption 2.3. Notice that

$$
\left\|\int_{x_{i-1}}^{x_i}c_\pm(t,x)\theta_\pm(t,x)\mathrm{d}x\right\|^2\leq qnM_c^2\int_{x_{i-1}}^{x_i}\|\theta_\pm(t,x)\|^2\mathrm{d}x,
$$

where M_c is defined as in Assumption 2.4. It follows that

$$\begin{aligned}R_1 \geq \int_0^T \mathrm{e}^{-Lt}\sum_{i=1}^{l}\Bigg[&\frac{L-(\tilde{M}+1)M_0\tilde{M}-2nM_1\tilde{M}}{qnM_c^2}\left\|\int_{x_{i-1}}^{x_i} c_+(t,x)\theta_+(t,x)\mathrm{d}x\right\|^2\\ &+\frac{L-(\tilde{M}+1)M_0\tilde{M}-2nM_1\tilde{M}}{qnM_c^2}\left\|\int_{x_{i-1}}^{x_i} c_-(t,x)\theta_-(t,x)\mathrm{d}x\right\|^2\\ &-2\tilde{M}\|\omega(t)\|\left\|\int_{x_{i-1}}^{x_i} c_+(t,x)\theta_+(t,x)\mathrm{d}x\right\|-2\tilde{M}\|\omega(t)\|\left\|\int_{x_{i-1}}^{x_i} c_-(t,x)\theta_-(t,x)\mathrm{d}x\right\|\Bigg]\mathrm{d}t,\end{aligned}$$

provided that $L>(\tilde{M}+1)M_0\tilde{M}+2nM_1\tilde{M}$. Therefore, we conclude that

$$\begin{aligned}R_1+R_2 \geq \int_0^T \mathrm{e}^{-Lt}\Bigg\{&\frac{\varepsilon^*}{2}\|\omega(t)\|^2\\ &+\sum_{i=1}^{l}\Bigg[\frac{L-(\tilde{M}+1)M_0\tilde{M}-2nM_1\tilde{M}}{qnM_c^2}\left\|\int_{x_{i-1}}^{x_i} c_+(t,x)\theta_+(t,x)\mathrm{d}x\right\|^2\\ &-2\tilde{M}\|\omega(t)\|\left\|\int_{x_{i-1}}^{x_i} c_+(t,x)\theta_+(t,x)\mathrm{d}x\right\|+\frac{\varepsilon^*}{4l}\|\omega(t)\|^2\\ &+\frac{L-(\tilde{M}+1)M_0\tilde{M}-2nM_1\tilde{M}}{qnM_c^2}\left\|\int_{x_{i-1}}^{x_i} c_-(t,x)\theta_-(t,x)\mathrm{d}x\right\|^2\\ &-2\tilde{M}\|\omega(t)\|\left\|\int_{x_{i-1}}^{x_i} c_-(t,x)\theta_-(t,x)\mathrm{d}x\right\|+\frac{\varepsilon^*}{4l}\|\omega(t)\|^2\Bigg]\Bigg\}\mathrm{d}t,\end{aligned}$$

provided that $L>(\tilde{M}+1)M_0\tilde{M}+2nM_1\tilde{M}$. Then we choose that

$$L \geq \frac{4lqnM_c^2\tilde{M}^2}{\varepsilon^*}+(\tilde{M}+1)M_0\tilde{M}+2nM_1\tilde{M}. \tag{4.32}$$

Consequently, we obtain that

$$R_1+R_2 \geq \int_0^T \frac{\varepsilon^*}{2}\mathrm{e}^{-Lt}\|\omega(t)\|^2\mathrm{d}t \geq \frac{\varepsilon^*}{2}\mathrm{e}^{-LT}\int_0^T \|\omega(t)\|^2\mathrm{d}t. \tag{4.33}$$

Together with Claim B.3, this concludes the proof of Lemma 4.10 with $c_T = \frac{\varepsilon^*}{2\tilde{M}}\mathrm{e}^{-LT}$. ∎

Appendix A. Broad solutions. We consider the following hyperbolic system, which includes all the systems of this paper. For (t,x) in $\mathcal{D}_l(t_0)$,

$$\partial_t w(t,x)+\Lambda(t,x)\partial_x w(t,x)=A(t,x)w(t,x)+J(t,x), \tag{A.1a}$$

$$w_+(t,x_i^+)=w_+(t,x_i^-)+\sigma^{i+}(t),\quad i=1,\ldots,l-1, \tag{A.1b}$$

$$w_+(t,0)=Q(t)w_-(t,0)+\sigma^{0+}(t), \tag{A.1c}$$

$$w_-(t,x_i^-)=w_-(t,x_i^+)+\sigma^{i-}(t),\quad i=1,\ldots,l-1, \tag{A.1d}$$

$$w_-(t,1)=\int_0^1 L(t,\xi)w(t,\xi)\mathrm{d}\xi+\sigma^{l-}(t), \tag{A.1e}$$

$$w(t_0,x)=w^0(x), \tag{A.1f}$$

where $w(t,x)$ in $\mathbb{R}^n$ is the state, and w^0 in $L^2(0,1)^n$ is the initial data. Functions J in $L^2(\mathcal{D}(0))^n$, $\sigma^- := (\sigma^{1-},\dots,\sigma^{l-})$ in $L^2(0,T)^{m\times l}$, and $\sigma^+ := (\sigma^{0+},\dots,\sigma^{(l-1)+})$ in $L^2(0,T)^{p\times l}$ are the nonhomogeneous terms. For the coefficients involved in system (A.1), let us make the following assumptions.

Assumption A.1. Assume that Λ, A, and Q satisfy Assumptions 2.1 to 2.3 and that L is in $L^\infty(\mathcal{D}(0))^{m\times n}$ satisfying $\|L\|_{L^\infty(\mathcal{D}(0))^{m\times n}} \le M_1$ for M_1 defined as in Assumption 2.3.

Notice that L is defined over time interval $(0,T)$. The reason lies in the regularity of the feedback gain functions in this paper.

A.1. Definition of broad solution. Let us now introduce the definition of broad solution or so-called solution along the characteristics. This definition is similar to the definition of broad solution in [12]. Recalling the notation in section 3, we introduce $\bar{s}_j^{\text{in}}(t_0;t,x) = \max\{t_0, s_j^{\text{in}}(t,x)\}$ for $j=1,\dots,n$, and

$$i(x) = i, \quad \text{if } x \in (x_{i-1},x_i), \quad i=1,\dots,l.$$

Similar to the methods used in [12], integrating the jth equation in (A.1a) along the characteristic $\chi_j(s;t,x)$ and applying appropriate boundary, jump, or initial conditions, we obtain the following system of integral equation. For (t,x) in $\mathcal{D}_l(t_0)$,

$$\text{(A.2)}\quad \begin{aligned} w_j(t,x) = I_j(w)(t,x) &+ \int_{\bar{s}_j^{\text{in}}(t_0;t,x)}^{t} \sum_{k=1}^{n} a_{jk}(s,\chi_j(s;t,x))w_k(s,\chi_j(s;t,x))\mathrm{d}s \\ &+ \int_{\bar{s}_j^{\text{in}}(t_0;t,x)}^{t} J_j(s,\chi_j(s;t,x))\mathrm{d}s, \end{aligned}$$

where for $j=1,\dots,m$,

$$\text{(A.3)}\quad I_j(w)(t,x) = \begin{cases} \displaystyle\int_0^1 L_{j,:}(s_j^{\text{in}}(t,x),\xi)w(s_j^{\text{in}}(t,x),\xi)\mathrm{d}\xi + \sum_{k=i(x)}^{l} \sigma_j^{k-}(s_j^{\text{in}},k(t,x)), & \text{if } s_j^{\text{in}}(t,x) > t_0, \\ \displaystyle w_j^0(\chi_j(t_0;t,x)) + \sum_{k=i(x)}^{i(\chi_j(t_0;t,x))-1} \sigma_{j-m}^{k-}(s_j^{\text{in},k}(t,x)), & \text{if } s_j^{\text{in}}(t,x) < t_0, \end{cases}$$

and for $j=m+1,\dots,n$,

$$\text{(A.4)}\quad I_j(w)(t,x) = \begin{cases} \displaystyle Q_{j-m,:}(s_j^{\text{in}}(t,x))w_-(s_j^{\text{in}}(t,x),0) + \sum_{k=1}^{i(x)} \sigma_{j-m}^{(k-1)+}(s_j^{\text{in}},k(t,x)), & \text{if } s_j^{\text{in}}(t,x) > t_0, \\ \displaystyle w_j^0(\chi_j(t_0;t,x)) + \sum_{k=1+i(\chi_j(t_0;t,x))}^{i(x)} \sigma_{j-m}^{(k-1)+}(s_j^{\text{in},k}(t,x)), & \text{if } s_j^{\text{in}}(t,x) < t_0. \end{cases}$$

This leads to the following definition of the broad solution to system (A.1) over (t,x) in $\mathcal{D}_l(t_0)$.

DEFINITION A.2. *Let $T>0$, $0\le t_0<T$, w^0 in $L^2(0,1)^n$, J in $L^2(\mathcal{D}(0))^n$, σ^- in $L^2(0,T)^{m\times l}$, and σ^+ in $L^2(0,T)^{p\times l}$ be fixed. We say that w is the broad solution to system* (A.1) *over $\mathcal{D}_l(t_0)$ if w is in $\mathcal{B}_l(t_0)^n$ and if the integral equation* (A.2) *is satisfied for $j=1,\dots,n$, for a.e. $t_0<t<T$ and a.e. x in $(0,1)$.*

A.2. Well-posedness. In this section, the well-posedness result is provided.

THEOREM A.3. *Let $T>0$. Under Assumption* A.1, *for every $0\le t_0<T$, w^0 in $L^2(0,1)^n$, J in $L^2(\mathcal{D}(0))^n$, σ^- in $L^2(0,T)^{m\times l}$, and σ^+ in $L^2(0,T)^{p\times l}$, there exists a unique broad solution w in $\mathcal{B}_l(t_0)^n$ to* (A.1) *over $\mathcal{D}_l(t_0)$. Moreover, there exists $C=C(T)>0$ such that, for every $0\le t_0<T$, w^0 in $L^2(0,1)^n$, J in $L^2(\mathcal{D}(0))^n$, σ^- in $L^2(0,T)^{m\times l}$, and σ^+ in $L^2(0,T)^{p\times l}$, the broad solution w satisfies*

$$\begin{aligned}(\mathrm{A.5})\qquad &\|w\|_{L^\infty((t_0,T);L^2(0,1)^n)}+\|w\|_{L^\infty((0,1);L^2(t_0,T)^n)}\\ &\quad\le C(\|w^0\|_{L^2(0,1)^n}+\|J\|_{L^2(\mathcal{D}(0))^n}+\|\sigma^-\|_{L^2(0,T)^{m\times l}}+\|\sigma^+\|_{L^2(0,T)^{p\times l}}).\end{aligned}$$

The proof is based on the proof of Theorem A.2 of [12]. We provide only a sketch of the proof here, highlighting the differences from the proof of Theorem A.2 presented in [12].

Sketch of the proof of Theorem A.3. The basic idea is the following fixed point method. A function $w:\mathcal{D}(t_0)\to\mathbb{R}^n$ satisfies the integral equations (A.2) for a.e. $t_0<t<T$ and a.e. x in $(0,1)$ if and only if it is a fixed point of the map $\mathcal{A}:\mathcal{B}_l(t_0)^n\to\mathcal{B}_l(t_0)^n$ and $(\mathcal{A}(w))_j(t,x)$ is given by the expression on the right-hand side of (A.2). Let us now make $\mathcal{B}_l(t_0)^n$ a Banach space by equipping it with the weighted norm $\|w\|_{\mathcal{B}_l(t_0)^n}=\|w\|_{\mathcal{B}_1}+\|w\|_{\mathcal{B}_2}$, where

$$\|w\|_{\mathcal{B}_1}=\max_{t\in[t_0,T]}\mathrm{e}^{-\frac{L_1}{2}(t-t_0)}\sqrt{\int_0^1\sum_{j=1}^n|w_j(t,x)|^2\mathrm{e}^{-L_2x}\mathrm{d}x},$$

$$\|w\|_{\mathcal{B}_2}=\max_{x\in[0,1]}\mathrm{e}^{\frac{L_2}{2}(1-x)}\sqrt{\int_{t_0}^T\sum_{j=1}^n|w_j(t,x)|^2\mathrm{e}^{-L_1(t-t_0)}\mathrm{d}t},$$

where $L_1,L_2>0$ are constants independent of T, t_0, w_0, σ, and J that will be fixed below. The similar weight norms are also used in [13, 15]. Our goal is to show that, for $L_1,L_2>0$ large enough,

$$(\mathrm{A.6})\qquad \|\mathcal{A}(w^1)-\mathcal{A}(w^2)\|_{\mathcal{B}_l(t_0)^n}\le\frac{1}{2}\|w^1-w^2\|_{\mathcal{B}_l(t_0)^n}\quad\forall w^1,w^2\in\mathcal{B}_l(t_0)^n.$$

Actually, the proof of (A.6) is the same as in [12]. Indeed, we introduce $w:=w^1-w^2$, so that $\mathcal{A}(w^1)-\mathcal{A}(w^2)$ is equal to the right-hand side of (A.2) with $w^0=0$, $J=0$, $\sigma^+=0$, and $\sigma^-=0$. This is a special case in [12]. Therefore, (A.6) is established by following the proof in [12]. The remaining task is to verify that the estimate (A.5) holds. Indeed, using (A.6) we obtain that the fixed point w of $\mathcal{A}$ satisfies

$$(\mathrm{A.7})\qquad \frac{1}{2}\|w\|_{\mathcal{B}_l(t_0)^n}\le\|\mathcal{A}(0)\|_{\mathcal{B}_l(t_0)^n},$$

and straightforward computations show that

$$(\mathrm{A.8})\quad \|w\|^2_{L^\infty((t_0,T);L^2(0,1)^n)}\le\mathrm{e}^{L_2}\mathrm{e}^{L_1h}\|w\|^2_{\mathcal{B}_1},\quad \|w\|^2_{L^\infty((0,1);L^2(t_0,T)^n)}\le\mathrm{e}^{L_1h}\|w\|^2_{\mathcal{B}_2}.$$

Then the fixed point of $\mathcal{A}$ satisfies the estimate (A.5) if the right-hand side of (A.5) is the upper bound of $\|\mathcal{A}(0)\|_{\mathcal{B}_l(t_0)^n}$. By using changes of coordinate, (3.3), and (3.8), we obtain the following estimates:

$$\int_0^1 \sum_{j=1}^n \left| \int_{\bar{s}_j^{\text{in}}(t_0;t,x)}^t J_j(s,\chi_j(s;t,x))\mathrm{d}s \right|^2 \mathrm{e}^{-L_2 x}\mathrm{d}x \le \frac{\mathrm{e}^{M_0 T}}{\varepsilon_0}\|J\|^2_{L^2(\mathcal{D}(0))^n}, \tag{A.9}$$

$$\int_{t_0}^T \sum_{j=1}^n \left| \int_{\bar{s}_j^{\text{in}}(t_0;t,x)}^t J_j(s,\chi_j(s;t,x))\mathrm{d}s \right|^2 \mathrm{e}^{-L_1(t-t_0)}\mathrm{d}t \le \frac{\mathrm{e}^{M_0 T}}{\varepsilon_0^2}\|J\|^2_{L^2(\mathcal{D}(0))^n}, \tag{A.10}$$

$$\begin{aligned}&\int_0^1 \sum_{j=1}^n |I_j(0)(t,x)|^2 \mathrm{e}^{-L_2 x}\mathrm{d}x \\ &\quad\le 2\mathrm{e}^{M_0 T}(M_0\|\sigma^-\|^2_{L^2(0,T)^{m\times l}} + M_0\|\sigma^+\|^2_{L^2(0,T)^{p\times l}} + \|w^0\|^2_{L^2(0,1)^n}),\end{aligned} \tag{A.11}$$

$$\begin{aligned}&\int_{t_0}^T \sum_{j=1}^n |I_j(0)(t,x)|^2 \mathrm{e}^{-L_1(t-t_0)}\mathrm{d}t \\ &\quad\le 2\frac{\mathrm{e}^{M_0 T}}{\varepsilon_0}(M_0\|\sigma^-\|^2_{L^2(0,T)^{m\times l}} + M_0\|\sigma^+\|^2_{L^2(0,T)^{p\times l}} + \|w^0\|^2_{L^2(0,1)^n}),\end{aligned} \tag{A.12}$$

where ε_0 is defined as in Assumption 2.2. It follows from (A.9) and (A.11) that

$$\begin{aligned}\|\mathcal{A}(0)\|^2_{\mathcal{B}_1} \le 2\mathrm{e}^{M_0 T}\bigg(&2\|w^0\|^2_{L^2(0,1)^n} + \frac{1}{\varepsilon_0}\|J\|^2_{L^2(\mathcal{D}(0))^n} \\ &+ 2M_0\|\sigma^-\|^2_{L^2(0,T)^{m\times l}} + 2M_0\|\sigma^+\|^2_{L^2(0,T)^{p\times l}}\bigg).\end{aligned} \tag{A.13}$$

Similarly, from (A.10) and (A.12) we obtain that

$$\begin{aligned}\|\mathcal{A}(0)\|^2_{\mathcal{B}_2} \le 2\frac{\mathrm{e}^{M_0 T + L_2}}{\varepsilon_0}\bigg(&2\|w^0\|^2_{L^2(0,1)^n} + \frac{1}{\varepsilon_0}\|J\|^2_{L^2(\mathcal{D}(0))^n} \\ &+ 2M_0\|\sigma^-\|^2_{L^2(0,T)^{m\times l}} + 2M_0\|\sigma^+\|^2_{L^2(0,T)^{p\times l}}\bigg).\end{aligned} \tag{A.14}$$

Then the estimate (A.5) for the fixed point of $\mathcal{A}$ follows from (A.7), (A.8), (A.13), and (A.14). ◻

Appendix B. C^1 solutions. In this section, we show that the broad solution is also a C^1 solution if the data of the system are smooth enough. Moreover, the continuous dependence of the broad solutions on the system data is given. In the proofs of Lemmas 4.9 and 4.10, a C^1 solution is needed. Let us make the following assumptions for the coefficients involved in system (A.1).

Assumption B.1. Assume that Λ, A, Q, and L satisfy Assumption A.1, and that Λ, A, Q, and L are in $C^2(\overline{\mathcal{D}(0)})^{n\times n}$, $C^1(\overline{\mathcal{D}(0)})^{n\times n}$, $C^1([0,T])^{p\times m}$, and $C^1(\overline{\mathcal{D}(0)})^{m\times n}$, respectively.

The C^1 solution is given by the following theorem.

THEOREM B.2. *Let $T>0$. Under Assumption* B.1*, for every $0\le t_0<T$, w^0 in $C^1_U(\cup_{i=1}^l(x_{i-1},x_i))^n$, J in $C^1_U(\mathcal{D}_l(0))^n$, σ^- in $C^1([0,T])^{m\times l}$, and σ^+ in $C^1([0,T])^{p\times l}$ satisfying the compatibility conditions*

(B.1)
$$\sigma^{i\pm}(t_0) = w^0_\pm(x_i^\pm) - w^0_\pm(x_i^\mp), \quad i=1,\dots,l-1,$$
$$\sigma^{0+}(t_0) = w^0_+(0) - Q(t_0)w^0_-(0), \quad \sigma^{l-}(t_0) = w^0_-(1) - \int_0^1 L(t_0,\xi)w^0(\xi)\mathrm{d}\xi,$$
$$(\sigma^{i\pm})'(t_0) = -\Lambda_{\pm\pm}(t_0,x_i)((w^0_\pm)'(x_i^\pm) - (w^0_\pm)'(x_i^\mp)) + A_{\pm\pm}(t_0,x_i)\sigma^{i\pm}(t_0) - A_{\pm\mp}(t_0,x_i)\sigma^{i\mp}(t_0) + J_\pm(t_0,x_i^\pm) - J_\pm(t_0,x_i^\mp), \quad i=1,\dots,l-1,$$
$$(\sigma^{0+})'(t_0) = J_+(t_0,0) - Q(t_0)J_-(t_0,0) - \Lambda_{++}(t_0,0)(w^0_+)'(0) + Q(t_0)\Lambda_{--}(t_0,0)(w^0_-)'(0) + (A_{++}(t_0,0) - Q(t_0)A_{-+}(t_0,0))w^0_+(0) + (A_{+-}(t_0,0) - Q'(t_0) - Q(t_0)A_{--}(t_0,0))w^0_-(0),$$
$$(\sigma^{l-})'(t_0) = -\Lambda_{--}(t_0,1)(w^0_-)'(1) + A_{-+}(t_0,1)w^0_+(1) + A_{--}(t_0,1)w^0_-(1) + J_-(t_0,1) - \int_0^1 [L(t_0,\xi)(-\Lambda(t_0,\xi)(w^0)'(\xi) + A(t_0,\xi)w^0(\xi) + J(t_0,\xi)) + \partial_t L(t_0,\xi)w^0(\xi)]\mathrm{d}\xi,$$

there exists a unique solution w in $C^1_U(\mathcal{D}_l(t_0))^n$ to (A.1).

The proof follows the method in [13, Lemma 3.2] and [14, Lemma 2.1]. Here, we only provide a sketch of the proof, explaining how we apply the method from [13, Lemma 3.2] and [14, Lemma 2.1].

Sketch of the proof of Theorem B.2. Set, for u in $C^0_U(\mathcal{D}_l(t_0))^n$,

$$\|u\|_0 := \max_{1\le i\le n} \max_{(t,x)\in\mathcal{D}_l(t_0)} |\mathrm{e}^{-L_1 t - L_2 x} u_i(t,x)|,$$

and for u in $C^1_U(\mathcal{D}_l(t_0))^n$,

$$\|u\|_1 := \max\{\|u\|_0, \|\partial_t u\|_0, \|\partial_x u\|_0\},$$

where L_1 and L_2 are two large, positive constants determined later. Set

$$\mathcal{O} := \{v \in C^1_U(\mathcal{D}_l(t_0))^n | v(t_0,\cdot) = w^0,\, \partial_t v(t_0,\cdot) = -\Lambda(t_0,\cdot)(w^0)' + A(t_0,\cdot)w^0 + J(t_0,\cdot)\}.$$

For v in $\mathcal{O}$, let $w = \mathcal{A}_1(v)$ be defined as follows: for $j = 1,\dots,m$,

$$w_j(t,x) = I_j(v)(t,x) + \int_{\bar{s}_j^{\mathrm{in}}(t_0;t,x)}^t \sum_{k=1}^n a_{jk}(s,\chi_j(s;t,x))v_k(s,\chi_j(s;t,x))\mathrm{d}s + \int_{\bar{s}_j^{\mathrm{in}}(t_0;t,x)}^t J_j(s,\chi_j(s;t,x))\mathrm{d}s, \tag{B.2}$$

where $I_j(v)(t,x)$ is defined as in (A.3), and for $j = m+1,\dots,n$,

$$w_j(t,x) = I_j(w)(t,x) + \int_{\bar{s}_j^{\mathrm{in}}(t_0;t,x)}^t \sum_{k=1}^n a_{jk}(s,\chi_j(s;t,x))v_k(s,\chi_j(s;t,x))\mathrm{d}s + \int_{\bar{s}_j^{\mathrm{in}}(t_0;t,x)}^t J_j(s,\chi_j(s;t,x))\mathrm{d}s, \tag{B.3}$$

where $I_j(w)(t,x)$ is defined as in (A.4). Notice that for $j = m+1,\dots,n$, $I_j(w)(t,x)$ is only involved with w_-, which is defined by (B.2). It follows from Assumption B.1 and the compatibility conditions (B.1) that $\mathcal{A}_1(\mathcal{O}) \subset \mathcal{O}$. Direct calculation shows that the fixed point of $\mathcal{A}_1$ is the C^1 solution to (A.1). Our aim is to show that, for L_1 and L_2 large enough,

$$\|\mathcal{A}_1(v^1)-\mathcal{A}_1(v^2)\|_1 \le \frac{1}{2}\|v^1-v^2\|_1 \quad \forall v^1, v^2 \in \mathcal{O}. \tag{B.4}$$

We can directly use the method from [13, Lemma 3.2] and [14, Lemma 2.1] to prove (B.4), since (A.1) is linear. Indeed, we introduce $v := v^1 - v^2$, so that $w := \mathcal{A}_1(v^1) - \mathcal{A}_1(v^2)$ is equal to the right-hand sides of (B.2) and (B.3) with $w^0 = 0$, $J = 0$, $\sigma^+ = 0$, and $\sigma^- = 0$. This is a special case in [14, Lemma 2.1]. Therefore, (B.4) is established by following the proof in [14, Lemma 2.1]. ∎

As for the continuous dependence of the broad solutions on the system data, one can prove the following claim by using the same method as in [9, Theorem 3.5].

Claim B.3. *For Λ, A, Q, and L satisfying Assumption* A.1*, and w^0 in $L^2(0,1)^n$, J in $L^2(\mathcal{D}(0))^n$, σ^- in $L^2(0,T)^{m\times l}$, and σ^+ in $L^2(0,T)^{p\times l}$, let w in $\mathcal{B}_l(t_0)^n$ be the broad solution to system* (A.1) *over $\mathcal{D}_l(t_0)$. For $k \ge 1$, Λ^k, A^k, Q^k, and L^k satisfying Assumption* B.1*, and $w^{0,k}$ in $C^1_U(\cup_{i=1}^l (x_{i-1}, x_i))^n$, J^k in $C^1_U(\mathcal{D}_l(0))^n$, $\sigma^{-,k}$ in $C^1([0,T])^{m\times l}$, and $\sigma^{+,k}$ in $C^1([0,T])^{p\times l}$ satisfying the compatibility conditions* (B.1)*, let w^k in $C^1_U(\mathcal{D}_l(t_0))^n$ be the C^1 solution to system* (A.1)*. Assume that*

$$\begin{aligned}&(\Lambda^k, A^k, Q^k, L^k, w^{0,k}, J^k, \sigma^{-,k}, \sigma^{+,k}) \to (\Lambda, A, Q, L, w^0, J, \sigma^-, \sigma^+) \quad \textit{in}\\ &C^1(\overline{\mathcal{D}(0)})^{n\times n} \times C^0(\overline{\mathcal{D}(0)})^{n\times n} \times C^0([0,T])^{p\times m} \times L^\infty(\mathcal{D}(0))^{m\times n}\\ &\quad \times L^2(0,1)^n \times L^2(\mathcal{D}(0))^n \times L^2(0,T)^{m\times l} \times L^2(0,T)^{p\times l} \quad \textit{as} \quad k\to\infty.\end{aligned}$$

Then we have $w^k \to w$ in $\mathcal{B}_l(t_0)^n$ as $k \to \infty$.

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